

MODELING OF FORWARD CONTRACTS IN  
ELECTRICITY MARKETS

BY

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## Summary

Since the deregulation of electricity, new models were needed to quantify the price of contracts with specified delivery period of electricity. In this thesis we have looked at two such models. The first model is given in [6], here the electricity price is modeled as a sum of log-normal forwards. The main concern was how the swap price and log-returns of the electricity price would behave, given the log-normal forwards.

Secondly we compare the above model for electricity prices, with a second model for electricity prices given by [9]. The comparison is interesting, since the second model makes an approximation to the electricity price which is not consistent with what the mathematics tell us.

And finally we simulate a call option price with each model as underlying.

We have observed differences in the swap price paths and log-returns estimated by the two models. Because of the difference price paths, the price of the call option gave different values.



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# Chapter 1

## Introduction

In the early 1990s the electricity market was liberalized, Norway was the first to liberalize it and established Nord Pool, the leading power exchange January 1th 1991. Shortly after, in 1992, Sweden joined the organized market. The new market gave an opening to a new area, of both economical and mathematical sense. First of all, giving speculators a new area to trade, and also pension and hedge funds a new place to invest. For mathematicians, the challenge of finding realistic models, which best represents the traded products.

It is in this market we will focus, we will look into electricity contracts with different time to maturity. Three models for electricity contracts will be specified. Each represents a good approximation to the traded electricity contracts. First we will investigate the behavior of the electricity contract in which we will call the "real" price. Secondly we will look at a model which deliberately make an approximation not consistent with the mathematics. It is therefore interesting to look if the approximation differ significantly from one of the other models.

The remainder of this chapter gives an introduction to aspects in the electricity market in which we will be concentrating on. The important Norwegian power exchange, Nord Pool ASA. Influences on electricity prices and the modeling challenges. An introduction to derivatives which are of interest in this thesis, and the meaning of some financial Greeks. Chapter 2 gives an introduction to th models for the electricity contracts, chapter 3 provides numerical examples of the electricity contracts and call options with the electricity contract as underlying, before we conclude and refer to further research in chapter 4.

### 1.1 Electricity: Nord Pool Group, Influences and Modeling Challenges

#### Nord Pool Group [2]

Nord Pool is the power exchange in Scandinavia. It is divided in two separate divisions. Nord Pool ASA and Nord Pool Spot.

Nord Pool ASA are owned by Statnett (50%) and Svenska Kraftnät (50%). Nord Pool ASA trade standardized financial electricity derivatives, base and peak load futures and forwards, options and Contract of Difference. The difference in futures and forwards follows in the next section. The derivatives being traded are contracts with price denoted in EUR/MWh, the reference price for the nordic contracts is the System Price of the total Nordic power market, European Energy Exchange for German Power and Amsterdam Power Exchange for Dutch Power. The System Price is also denoted the unconstrained market clearing price since the trading capacities between the bidding areas have not been taken into account in finding this price.

Including in Nord Pool ASA are Nord Pool Clearing and Nord Pool Consulting, responsible for clearing all contracts traded at Nord Pool and contracts registered for clearing traded at the bilateral financial markets.

Nord Pool Spot AS is responsible for the physical-delivery spot each hour in the physical market for Norway, Sweden, Finland and Denmark. This spot price is the equilibrium of supply-demand in the market on a hour-to-hour basis. Established on the balance between bids and offers from all market participants.

## **Influences on electricity prices**

Since electricity supply mainly are made and driven by nature, electricity is highly influenced by factors out of human reach, but some sudden peaks may have a human touch, both in lowering the supply but manage to increase shortly after. The following will give the main contributions to electricity price fluctuations, given by [2].

- Temperature - e.g. low temperature in the winter season, increase the demand of electricity and hence the price increase.
- Precipitation
- Transmission capacity - e.g. increase in demand but capacity shortage could increase the price.
- Nord Pool is tied to the Russian, German and Polish market hence supply and demand there will influence the Nordic prices.
- Prices of other energy sources than water, such as coal, gas and nuclear energy, which is of great demand in Denmark, Sweden and Finland, influence the prices.
- Expanding or decreasing of generating capacity.
- Currency fluctuations.

The main and most important observation considering the electricity price fluctuations, is the influence given by temperature. Giving a prediction advantage

beyond the regular stock market, at least for long dated contracts.

Before we start defining a model, suited for the electricity market it is crucial to have access to empirical data. On Nord Pool Spot, historical daily spot prices are available.

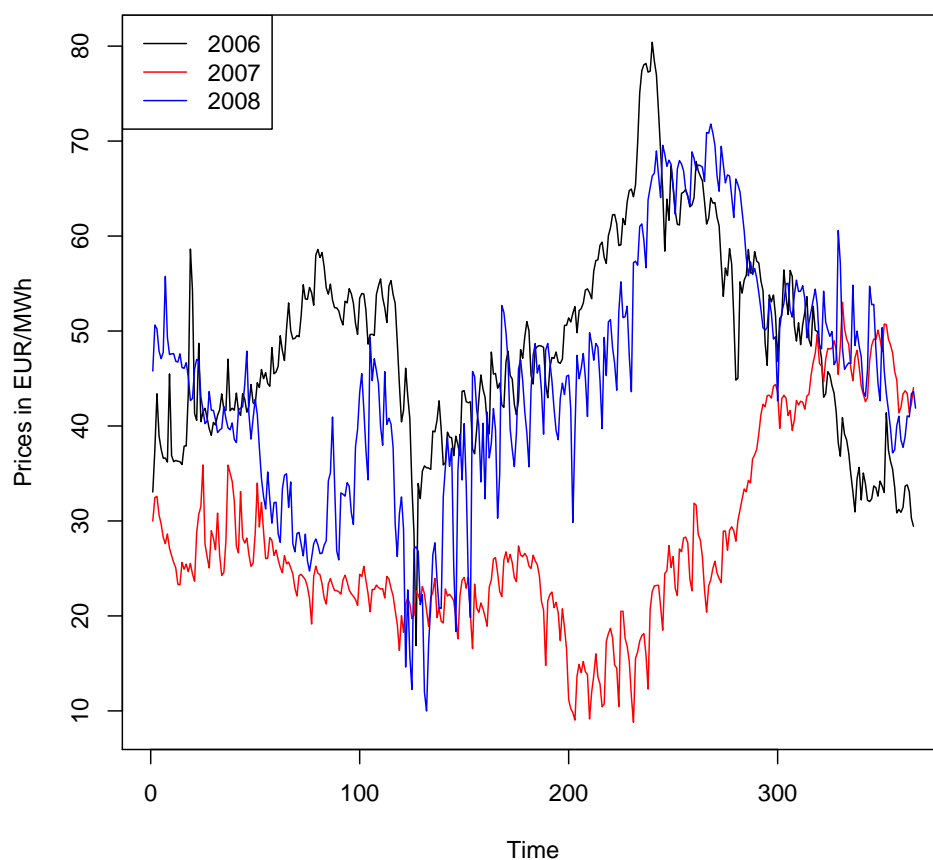


Figure 1.1: Daily Spot prices from Elspot in year 2006, 2007 and 2008

In figure 1.1 we have plotted daily spot prices, for year 2006, 2007 and 2008. There are similarities with year 2006 and 2008, they hit bottom around April and increase to a peak around August, with a decrease towards December. 2007 follow the same path, but with a price significant lower than 2006 and 2008. Hence there may have been non-financial factors affecting the prices, but similarity in the fluctuations indicates season dependencies.

## Modeling Challenges

In regular financial markets, the traded asset have the advantage of being store-able, without any extra cost to it. The purpose of waiting coincide with an expectation of an increase in the asset, giving you the possibility to sell it at a higher price. But this relies heavily on the possibility to store without any extra costs. For electricity this is not possible. This view will be considered throughly in chapter 2.

When working with financial assets the main problem is always, what to expect in the future. And how to be able to manage this risk involving with financial assets, how to estimate it and will the estimate be close to reality. Above we discussed the importance to look at historical data, and find models which justifies the structure which have been observed. The electricity spot prices seen above, had remarkable same features which implied seasonality. This again gives us indication that the model needed to be constructed can *not* be stationary. If we model the electricity prices without season dependencies, we will get a stationary process, *but* not a realistic one. In chapter 3, we model electricity prices, not depending on season for the purpose of similarity.

In addition we need to pay attention to another feature stated by Samuelson in 1965, short dated contracts will be affected by information revealed close to maturity of the contract, and hence the volatility of future price returns will increase. For long dated contracts this volatility effect will be wiped out. An example can give the right insight. Consider a contract with delivery of electricity in one year, you know there will be a summer with less demand of electricity than in the winter, hence you will pass through high and low price levels. On the other hand, consider a contract with delivery of electricity during a week in January, which is a cold month, but there can suddenly be a weather change or one of the supply station break down and decrease the supply, this influence the price significantly close to maturity. And will be seen as spikes in the spot price fluctuations.

The conclusion is therefore, the model needed to simulate electricity prices must contain season, maturity and spike effect. This will improve the model to better capture real fluctuations.

The previous section gave valuable insight to the behavior of electricity prices. Though there are factors beyond these, not easy captured by a model i.e. politics and better weather forecast than your opponents. These factors will not be discussed any further.

The following sections gives an introduction to a selection of contracts traded at Nord Pool and which will be referred to in later chapters.

## 1.2 Future Contract

A future contract in general, is a contract where two parties at time  $t > 0$ , agree for a price  $f(t, u)$  to be paid at a future specified time point  $u > t$ , in exchange for the agreed commodity. At Nord Pool they refer to a future contract with delivery of electricity in a period rather than a point. But except for that, the behavior is the same. That being, a future contract is a contract with mark-to-market settlement. Each day, after contract agreement, the mark-to-market settlement covers gains or losses from a day-to-day changes in the market price of each contract. Future contracts with delivery of electricity during a day or a week are traded at Nord Pool.

For example say if you buy a future contract at price 50 EUR/MWh, the next day the market value the contract to 55 EUR/MWh, then your account will be credit 5 EUR/MWh, and the seller will be debited 5 EUR/MWh, an important remark, the initial price of the contract 50 EUR/MWh are *not* delivered yet, this payment will be transferred at time  $u$ .

## 1.3 Forward Contract

As for the future contract a forward contract at Nord Pool has a delivery period, rather than a delivery point. There are an agreement of delivery of electricity in the future, but the price of the contract will *not* be settled mark-to-market. Therefore the difference in the price agreed at time  $t < u$ , with delivery at time  $u$  and the actual price of the contract at time of delivery  $u$ , need to be covered by the "loosing part" at time  $u$ . At Nord Pool, there exist forward contracts with delivery of electricity during a month, quarter or year.

Comparing future and forward contracts, the structure is the same, but the risk involving in a future contract is significant lower than with a forward contract.

## 1.4 Swap Contract

In a regular financial market a swap contract is an obligation between two parties to exchange some specified cash flows over a period in the future. Observe in the general case, there are no price attached to this contract. But only the specified cash flows between the two involved parties.

The nature of forward and future contracts described at Nord Pool resembles swap contract, because you swap between floating to fixed electricity price. Since we do not distinguish between future and forward contracts will we refer to future and forward contracts traded at Nord Pool for swap contracts through the thesis.

European options are the only options traded at Nord Pool, our focus will be on a call option.

## 1.5 Call Option

A call option  $P = \max(F(T) - K, 0)$ , is an option between two parties, where they at time  $t > 0$  agree for a price  $K$  referred to as the strike or strike price, that gives the buyer at time  $T > t$ , the option to buy e.g. a stock  $F$  for the price  $K$ .

Contrast to the future, forward and swap contracts, the buyer are not obliged to buy the stock, but have bought the right to do it. The reason for participate in an option, may be to reduce the risk that the stock becomes too expensive at future time  $T$ . In our context the stock will be replaced by a swap contract.

## 1.6 Greeks

The Greeks are a family of different measures, denoted by Greek letters. Each Greek models the sensitivity of the value of a portfolio to a small change in the underlying variables. For an option, the value is related to the underlying variables: price of the underlying, strike price, interest rate, time to expiration and volatility of the underlying. Each of these variables makes the value uncertain. An investor needs to know how much a change in one of the underlying parameters affects the value of the portfolio. Hence the Greeks measure the change in these parameters. Our Greek of interest will be the delta. Following will be an introduction to the delta and briefly the other Greeks and its interpretation.

### Delta

The delta  $\Delta$  describes how sensitive the option's value  $P$ , is to changes in the underlying derivative price  $F$ , in the mathematical sense,

$$\Delta = \frac{\partial P}{\partial F} \quad (1.1)$$

If the  $\Delta = 0$  the option is delta neutral, a change in the underlying derivative will not affect the price of the option. For a call option the  $\Delta \geq 0$ , e.g. if the delta is 0.4 the price of the call option will increase with 40% if the price of the underlying increases.

### Vega, Theta, Rho and Gamma

Vega Measures the change in an option's value due to changes in the volatility  $\sigma$ :  $\nu = \frac{\partial P}{\partial \sigma}$

Theta Measures the change in an option's value due to changes in time to expiration  $T$ :  $\Theta = \frac{\partial P}{\partial T}$

Rho Measures the change in an option's value due to changes in the interest rate  $r$ :  $\rho = \frac{\partial P}{\partial r}$

Gamma Measures the change in delta  $\Delta$  due to changes in the underlying derivative price  $F$ :  $\gamma = \frac{\partial \Delta}{\partial F}$

## Chapter 2

# Modeling Approaches

The following chapter will refer to general probability and measure theory. The needed definitions are given in Appendix A. Some of the definitions are too theoretical concerning what our main purpose in this thesis is, but are included for the purpose of holding the mathematics right and hopefully not having the need of extern literature by your side.

In the following, the meaning of a forward contract is in the general meaning, *not* given as contract with delivery of the agreed commodity over a period, but as a fixed delivery point.

First we introduce the standard no-arbitrage condition for electricity. Our main subject of interest are to model the swap price with the Heath, Jarrow and Morton-model (HJM). Given the HJM framework we introduce three different approaches for the swap price. The main purpose of each of these models, is whether a given log-normal forward dynamic, transforms to a log-normal swap dynamic.

### 2.1 Real World vs. Mathematical World

In the financial dynamic trading market, an important trading strategy is to hedge i.e. go short in one asset but long in another asset. In this way you minimize the risk involving in the first asset. In order to have the ability to participate in such a strategy, the traded assets need to be store-able. That is, the cost involving to store the traded commodity in which you went long in, can not be significant. For electricity the important feature is the absence of store-ability. At least for small participants that do not own a power plant. We do not have the possibility to store electricity, and we are therefore not being able to hedge in the same way as in the stock market, in which give rise to a market not being complete. In an incomplete market there exists several measures which estimates a risk-neutral price. But these prices need not to coincide, hence we have a market with a possibility of arbitrage.

In addition the trading at Nord Pool need to be highlighted. The participants

at Nord Pool can be involved in both Nord Pool ASA and Nord Pool Spot. The trade at Nord Pool ASA involves estimation in future prices. Many of the participants on Nord Pool are owners of power plants. This means they can increase the supply if the spot price is high, but decrease the supply if spot price is low. But at the same time participants can not sell electricity without water in the dikes. Therefore a good estimation of future prices are crucial. The best estimation of future prices involved to mention a few, estimation of the weather, temperature and supply and demand in other markets. Let us introduce an example. If a participant sell a forward contract, it must first and foremost deliver the agreed electricity. The seller, in order to earn money, hope the price of the contract is higher than the spot price at time of delivery. Cause then the seller has not lost profit, selling the forward contract instead of waiting and selling at spot. The buyer on the other hand, had another opinion of the future market and was able to hedge against the expected rise in spot price in the future.

## 2.2 Electricity Modeling Approaches

Throughout the thesis the following notation will be used. Let

- $F(t, T_1, T_2)$  = swap contract price at time  $t \leq T_1$ , with delivery period  $[T_1, T_2]$
- $f(t, u)$  = forward contract price at time  $t$ , with delivery time  $u \geq t$
- $\sigma(t, u)$  = volatility function at time  $t$ , with delivery time  $u \geq t$
- $S(t)$  = spot price at time  $t$
- $G(t)$  = log-return at time  $t$
- $W(t)$  = standard Brownian Motion under  $Q$
- $Q$  = the risk neutral probability measure

### The general framework [5]

Under the risk-neutral measure  $Q$  the price of the spot and the forward at time  $u$  must coincide in order for the no-arbitrage condition to hold under the given filtration  $\mathcal{F}_t$ , if not, we have an arbitrage possibility, that is:

$$f(t, u) = E_Q[S(u)|\mathcal{F}_t] \quad (2.1)$$

Let us show the connection between the forward - and swap price.

Remember a swap contract in the electricity market, is a continuous flow of electricity instead of a single delivery. In the mathematical sense can we look at time  $t$  value of the payoff, over the delivery period  $[T_1, T_2]$ , for electricity as the difference between the spot price  $S(t)$  and the swap price  $F(t, T_1, T_2)$  at time of delivery,

$$\int_{T_1}^{T_2} e^{-r(u-t)}(S(u) - F(t, T_1, T_2))du$$



Where  $e^{-r(u-t)}$  is the discount factor, with constant rate of interest  $r$ .

It is costless to enter a swap contract at time  $t$  because you pay at time of delivery, the difference between the spot and swap price during delivery of electricity need to be equal to zero under the risk-neutral measure  $Q$  in order for the no-arbitrage condition to hold,

$$e^{-rt} E_Q \left[ \int_{T_1}^{T_2} e^{-r(u-t)} (S(u) - F(t, T_1, T_2)) du | \mathcal{F}_t \right] = 0$$

The swap contract, is settled at time  $t$ , hence we can assume the swap price to be adapted.

$$F(t, T_1, T_2) = E_Q \left[ \int_{T_1}^{T_2} \frac{re^{-ru}}{e^{-rT_1} - e^{-rT_2}} S(u) du | \mathcal{F}_t \right]$$

We may assume that the contract settles at maturity  $T_2$ :

$$e^{-rT_2} E_Q \left[ \int_{T_1}^{T_2} (S(u) - F(t, T_1, T_2)) du | \mathcal{F}_t \right] = 0$$

Which finally yields the price of a swap contract is an average of spot prices during the delivery period.

$$F(t, T_1, T_2) = E_Q \left[ \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} S(u) du | \mathcal{F}_t \right]$$

Now we can state the relationship for swap the swap  $F(t, T_1, T_2)$  and forward  $f(t, u)$  price:

**Proposition 1.** [5]

Suppose  $E_Q \left[ \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} S(u) du \right] < \infty$ , it holds that

$$F(t, T_1, T_2) = \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(t, u) du \quad (2.2)$$

Hence, given we work under the no-arbitrage condition, a swap price is nothing but a continuous flow of forwards.

In summation we can either start with a spot price model given by (2.1) and derive the forward price. This approach have been discussed in papers, to mention a few, Burger et al. (2004) [3] and Erlwein et al. (2007) [4]. Or directly estimate the forward price. It is this path we will continue on, following under.

The following sections will be our main theory. First we introduce the given framework for our model, the Heath, Jarrow and Morton model. Second we specify a log-normal forward model, and establish a mathematical function

for the swap price contract. Thirdly we introduce three different models each having the same goal, that is to find the best swap-price under the HJM-model which is mathematical right and at the same time a good approximation to empirical data.

## HJM-model

The Heath, Jarrow and Morton model was proposed by David Heath, Robert A. Jarrow and Andrew Morton in 1992. First and foremost it is a model for the interest rates dynamic in an arbitrage-free framework. Given the arbitrage-free assumption, they establish the drift function, which will be fully dependent on the choice of the volatility function in order for a risk-neutral-probability measure  $Q$  to exist.

Let  $f(t, u)$  be the forward interest rate at time  $t$  with delivery at time  $u$  given as

$$\begin{aligned} df(t, u) &= \alpha(t, u)dt + \sigma(t, u)^T dW(t) \\ f(0, u) &= f^M(0, T) \end{aligned}$$

Where  $u \rightarrow f^M(0, T)$  is the markets instantaneous-forward curve at time  $t = 0$  and where  $W = (W_1, W_2, \dots, W_N)$  is an  $N$ -dimensional Brownian Motion,  $\sigma(t, u) = (\sigma_1(t, u), \sigma_2(t, u), \dots, \sigma_N(t, u))$  is an vector of adapted processes and  $\alpha(t, u)$  scalar product between the two vectors  $\sigma(t, u)$  and  $dW(t)$ .

Following the procedure given by [1], under a no-arbitrage condition the drift function becomes

$$\begin{aligned} \alpha(t, u) &= \sigma(t, u) \int_t^T \sigma(t, s)ds \\ &= \sum_{i=1}^N \sigma_i(t, u) \int_t^T \sigma_i(t, s)ds \end{aligned}$$

Giving the following forward rate

$$f(t, u) = f(0, u) + \sum_{i=1}^N \int_0^t \sigma_i(k, u) \int_k^t \sigma_i(k, s)dk du + \sum_{i=1}^N \int_0^t \sigma_i(s, u)dW_i(s)$$

In order to have a Markovian forward rate, the volatility function must be separate-able, this was proven by Carverhill (1994) i.e.

$$\sigma_i(t, u) = \sigma_{i,1}(t)\sigma_{i,2}(u) \quad \forall i$$

Adapting this to our context, the HJM-model will be used to model the forward and swap price. The HJM-model is highly favorable because we get the dynamics for the whole future price curve, and we can consider the market to be complete which gives rise to a price under a risk-neutral-probability. But the disadvantage is that future prices do not reveal information about spot prices. As discussed in chapter 1, the spot price is important because it is the leading reference when pricing electric derivatives.

## Pricing of swaps via HJM-model

In the above section we derived the forward price from the spot and further the swap price. Here we start out with models for the swap and forward ignoring the spot price entirely, with respect to the HJM-model.

Let  $(\Omega, \mathcal{F}, \mathcal{F}_{t \in [0, T]}, Q)$  be a complete probability space. We assume our market consists of swap contracts  $F(t, T_1, T_2)$  with disjoint delivery periods, and with price dynamic for the swap contracts to be

$$dF(t, T_1, T_2) = F(t, T_1, T_2)\Sigma(t, T_1, T_2)dW(t) \quad (2.3)$$

Where  $\Sigma$  satisfies the conditions given in Appendix A, such that the swap dynamic becomes a square-integrable martingale.

Let the forward dynamic  $f(t, u)$  for  $0 \leq t \leq u \leq T$  be

$$df(t, u) = f(t, u)\sigma(t, u)dW(t) \quad (2.4)$$

Where  $\sigma$  satisfies the conditions given in Appendix A, such that the forward dynamic become a square-integrable martingale.

Hence the swap and forward dynamic, both are martingales under the risk-neutral probability  $Q$ , and are driven by a single Brownian Motion  $W(t)$ .

We have chosen to let both the swap and forward dynamic to be log-normal models. Log-normal models are easy to work with. And since we want to find the price of electricity, then a model, not being able to give us negative prices will be favorable.

In the following, we only consider settlement at maturity of the contract. This means the buyer pays at time of delivery, rather than at a continuous flow during the delivery period. For more details see Appendix A.

Under the risk-neutral-probability measure  $Q$ , there are no possibility in finding an arbitrage possibility. This gives rise to the following lemma.

**Lemma 1.** [6]

*Consider a swap with delivery over the period  $[T_1, T_N]$  and  $N - 1$  swaps with delivery over the disjoint periods  $[T_i, T_{i+1}]$ ,  $i = 1, \dots, N-1$  and  $T_i < T_{i+1}$  where the union of these intervals coincides with  $[T_1, T_N]$ . Then the following holds:*

$$F(t, T_1, T_N) = \sum_{i=1}^{N-1} \frac{T_{i+1} - T_i}{T_N - T_1} F(t, T_i, T_{i+1}) \quad (2.5)$$

Under the conditions for HJM-models, (2.5) must hold for arbitrary delivery periods in order for the no-arbitrage to hold. We can make a mathematical approximation to 2.5 rather than the real life approach given above. This meaning,

what if there exist infinite contracts, each with different time to maturity, what will the sum converge to. The following will give a rough proof, approximating (2.5) to the continuous case.

[5] Let  $T_k = T_1 + (k + 1) \times \delta$ , where  $\delta = \frac{T_2 - T_1}{N}$ , letting  $N \rightarrow \infty$  and (2.5), leads us to

$$F(t, T_1, T_2) = \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} F(t, u, u) du \quad (2.6)$$

The purpose of all this is to establish a connection between the forward- and swap contract. The next lemma will lead the way,

**Lemma 2.** [6]

For  $0 \leq t \leq T_1$ , it holds that

$$F(t, T_1, T_1) = \lim_{T_2 \downarrow T_1} F(t, T_1, T_2) = f(t, T_1) = E_Q[S(T_1) | \mathcal{F}_t], \quad a.e. T_1 \in [t, T_1] \quad (2.7)$$

From a no mathematical view the last lemma is easy to swallow, the price of a swap contract, when delivery period becomes smaller and smaller, indicates a delivery point, hence the price of a forward with delivery at that point. With this lemma stated and (2.6) gives us the the well known continuous no-arbitrage condition of a swap contract, actually converging to a integral with the price of forward contracts  $f(t, u)$ , as integrand.

$$F(t, T_1, T_2) = \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} F(t, u, u) du \quad (2.8)$$

$$= \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(t, u) du \quad (2.9)$$

This section stated our model and the general no-arbitrage condition coincided with the one we have seen in section 2.2. In the following section we find the explicit expression for the swap price given a log-normal forward dynamic.

## The implied swap dynamics

**Theorem 1.** [6]

Assume the conditions given in Appendix A, part 5, holds for the coefficients of the forward dynamics (2.4). Then the coefficient function in the forward dynamics is related to the swap dynamics in the following way:

$$\Sigma(t, T_1, T_2) = \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} \sigma(t, u) f(t, u) du \quad (2.10)$$

Now we will express the swap contract, *given* the forward dynamic (2.4), remember (2.2) the strict continuous no-arbitrage condition:

$$F(t, T_1, T_2) = \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(t, u) du \quad (2.11)$$

$$= \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(0, u) du + \int_{T_1}^{T_2} \int_0^t \frac{1}{T_2 - T_1} \sigma(s, u) f(s, u) dW(s) du \quad (2.12)$$

$$= \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(0, u) du + \int_0^t \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} \sigma(s, u) f(s, u) du dW(s) \quad (2.13)$$

The last change of integrals, is justified using stochastic Fubini Theorem and classical Fubini-Tonelli Theorem (see Protter, 1990 p. 159-160). Also notice that the inner integral is  $\Sigma(s, T_1, T_2)$ , defined in Theorem 1. The inner integral can be decomposed using integration by parts, given by

$$\int h(u) g'(u) du = h(u) g(u) - \int h'(u) g(u) du \quad (2.14)$$

Here:

$$h(u) = \sigma(s, u) \quad (2.15)$$

$$g'(u) = \frac{1}{T_2 - T_1} f(s, u) \quad (2.16)$$

It follows:

$$\begin{aligned} \Sigma(s, T_1, T_2) &= \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} \sigma(s, u) f(s, u) du \\ &= \sigma(s, T_2) F(s, T_1, T_2) - \int_{T_1}^{T_2} \frac{\partial \sigma(s, u)}{\partial u} \int_{T_1}^u \frac{1}{T_2 - T_1} f(s, T) dT du \end{aligned} \quad (2.17)$$

The last integral is independent of  $T$ , and remember  $F(t, T_1, T_2) = \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(t, u) du$  we get the following:

$$\begin{aligned} \Sigma(s, T_1, T_2) &= \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} \sigma(s, u) f(s, u) du \\ &= \sigma(s, T_2) F(s, T_1, T_2) - \int_{T_1}^{T_2} \frac{\partial \sigma(s, u)}{\partial u} \frac{u - T_1}{T_2 - T_1} F(s, T_1, u) du \end{aligned} \quad (2.18)$$

And finally the swap can be expressed as follows:

$$\begin{aligned} F(t, T_1, T_2) &= \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(0, u) du + \int_0^t \sigma(s, T_2) F(s, T_1, T_2) \\ &\quad - \int_{T_1}^{T_2} \frac{\partial \sigma(s, u)}{\partial u} \frac{u - T_1}{T_2 - T_1} F(s, T_1, u) du dW(s) \end{aligned} \quad (2.19)$$

With dynamic given:

$$dF(t, T_1, T_2) = \sigma(t, T_2)F(t, T_1, T_2)dW(t) - \int_{T_1}^{T_2} \frac{\partial \sigma(t, u)}{\partial u} \frac{u - T_1}{T_2 - T_1} F(t, T_1, u) du dW(t) \quad (2.20)$$

From the final expression we notice that under the continuous and very strict no-arbitrage condition, the log-normal forward dynamic do not transfer the swap dynamic to be log-normal. The swap dynamic becomes non-Markovian, there are dependency from former swaps. In addition the swap returns will depend on the current state of the swap price, which are not independent over time increments. In order for the no-arbitrage condition to be fulfilled the volatility function can *not* depend on delivery time. For proof we refer the reader to [6] p. 1129.

From chapter 1, we discussed the important of having an electricity price model both depending on seasonality, maturity and with spikes, hence we need to do an approximation in order to make the model close to reality.

In the following we introduce three different models for a log-normal swap dynamic given the HJM-model. This implies we work under a no-arbitrage condition and trying to capture the stylized feature of electricity contracts, and at the same time being feasible.

### Bjerkssund, Rasmussen and Stensland Approach

The first approximation is the idea given by [9], the forward price dynamic at time  $t$  with delivery at time  $u$ ,  $t \leq u$  is given by

$$df(t, u) = f(t, u)\sigma(t, u)dW(t) \quad (2.21)$$

The volatility function is specified by  $\sigma(t, u) = \frac{a}{u-t+b} + c$  for positive constants  $a$ ,  $b$  and  $c$  estimated. Following the approach over, in order to have a log-normal swap dynamic, they approximate the estimated volatility function given by (2.18) as

$$\begin{aligned} \Sigma(s, T_1, T_2) &= \sigma(s, T_2)F(s, T_1, T_2) - \int_{T_1}^{T_2} \frac{\partial \sigma(s, u)}{\partial u} \frac{u - T_1}{T_2 - T_1} F(s, T_1, u) du \\ \frac{\Sigma(s, T_1, T_2)}{F(s, T_1, T_2)} &= \sigma(s, T_2) - \int_{T_1}^{T_2} \frac{\partial \sigma(s, u)}{\partial u} \frac{u - T_1}{T_2 - T_1} \frac{F(s, T_1, u)}{F(s, T_1, T_2)} du \\ &\approx \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} \sigma(s, u) du \end{aligned}$$

Giving rise to a log-normal swap price given by

$$F(t, T_1, T_2) \approx \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(0, u) du + \int_0^t \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} \sigma(s, u) du F(s, T_1, T_2) dW(s)$$

With swap dynamic given by

$$dF(t, T_1, T_2) = F(t, T_1, T_2)\Sigma(t, T_1, T_2)dW(t) \quad (2.22)$$

Given the specified volatility function  $\sigma(t, u) = \frac{a}{u-t+b} + c$ , we can calculate the volatility function for the swap dynamic  $\Sigma(t, T_1, T_2)$ , easy integration reveals

$$\Sigma(t, T_1, T_2) = \frac{a}{T_2 - T_1} \ln \left( \frac{T_2 - t + b}{T_1 - t + b} \right)$$

This approach ignore the dependencies of previous delivery time, in order to achieve the desired log-normal swap dynamic. Hence this approximation cuts out important mathematical facts, though a modified model, consulting [9] have been a good approximation to forward contracts traded at Nord Pool. In addition will we see the log-returns will become normal and independent not consistent with the above calculations.

### Benth and Koekebakker Approach

The second approach is the idea from [6]. The main object of discussion in this approach, is how to establish a rigorous model for the swap price without the need to approximate the given swap price model to a log-normal model, given we start with a log-normal forward dynamic. Hence they want to avoid the approximation method as in the approach given by Bjerksund, Rasmussen and Stensland as explained above. Following the given discussion in section 2.2, not being able to develop a log-normal swap dynamic given a log-normal forward dynamic, the following models are introduced.

1. The first is to establish the swap price directly under the continuous no-arbitrage condition given by (2.9) i.e.

$$F(t, T_1, T_2) = \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(t, u) du$$

$$df(t, u) = f(t, u)\sigma(t, u)dW(t)$$

With this approach you avoid the approximation given by Bjerksund et al., but at the same time you suppose there exists infinite forward contracts with different time to maturity. Since such forwards do not exists in this market we, need to establish a smoothing algorithm in order to get  $f(0, u)$ . Making an extra estimation procedure. A smoothing technique is throughout described in [5] chapter 7, Benth, Koekebakker and Ollmar (2005) [7] and Audet, Heiskanen, Keppo and Vehviläinen (2002) [8]. Since it do not exist infinite forwards, the approximation is not arbitrage free. Further, we can not establish an explicit expression for the log-returns or the price of a call option.

2. The second one is to model the swaps directly, i.e they avoid the forward dynamic, and instead model the swaps with no-overlapping delivery time in order for the no-arbitrage with finite swaps condition (2.5) to hold. An example will make it clear. Say you want the price of a swap with delivery period of one year, we decompose it to be the sum of monthly swaps, that is if we can not decompose the monthly swaps into swaps with shorter delivery time i.e. weekly swaps

$$F(t, T_1, T_{12}) = \sum_{i=1}^{11} \frac{T_{i+1} - T_i}{T_2 - T_1} F(t, T_i, T_{i+1}) \quad (2.23)$$

This is the only way we can have a log-normal model for the swaps and the no-arbitrage condition fulfilled, since we only look at trade able swaps. And not try to look at infinite many as in the continuous case. The idea is to make a family of "atomic swaps" consisting of the smallest "building blocks" that can not be decomposed by different swaps, and use *only* these to model swaps with longer delivery time.

With this approach, each swap dynamic are log-normal and you can have a volatility function depending on maturity, since we only work with no-overlapping swaps! Second we avoid the smoothing algorithm, since we can gather the swap prices directly from Nord Pool. The drawback however is the lack of connection to the spot. Remember section 2.2,  $f(t, u) = E_Q[S(u)|\mathcal{F}_t]$ , we do not have a model involving forward dynamics anymore. And the approach relies heavily that there do not exist swaps that overlap, from empirical data analyzed in Benth et al. [6] and given from Nord Pool, only 1793 of 54492 contracts were overlapping. Thirdly they avoid all other swaps with longer delivery time than the atomic swaps.

## 2.3 Summary

Given the HJM-framework we have tried to explain the problems when we want a no-arbitrage price. We have looked at three different models for the swap dynamic. Each having the goal in establishing a good and feasible model for the swap price given the no-arbitrage condition.

The model introduced by Bjerksund et al. have the drawback of approximating to a log-normal swap dynamic, ignoring dependencies from previous delivery points. And therefore not fulfill the no-arbitrage condition. But introduce a log-normal swap dynamic, which is positive when we want to price a option with a swap contract as underlying.



The second model, introduced by Benth et al., was the direct approach, here we must assume infinite forward contracts and we need to establish a smoothing technique.

The third model, introduced by Benth et al., was to model the swap price directly. We loose any connection to the spot price, but we do not need a smoothing algorithm in establishing the forward prices, since we can directly gather empirical data accessible at e.g. [www.nordpool.no](http://www.nordpool.no). In addition the volatility model can depend on maturity.

At last we can not underestimate the importance of a good volatility model. The volatility model need to capture increasing volatility as time-to-maturity decrease. Seasonality effect and spikes, each being important features in the electricity market. We will look at different volatility models in chapter 3.

Finally for a model not only being a piece of science, it must be able to derive the price fast and accurate. In the dynamic and fast trading environment, a simulation must be fast. The more stylished the models are, most certainly the time it will take to estimate the price will increase. Will the time compensate, and give a price more likely to be the best fit, or is it just a waste of time.



## Chapter 3

# Modeling of Electricity Contracts and a Call option

Based on the models given by Benth et al. and Bjerksund et al. in chapter 2, we derive explicit expressions for the price of the swap contracts. We simulate the contracts with different time to maturity and different delivery period of the contracts. Second we compare the two simulated contract models and highlight differences and what might have caused them. The main concerns are how the swap price model given by Benth et al. behave with different volatility functions and different time periods. And how does the model by Benth et al. approximate a log-normal swap dynamic. And finally will there be a significant difference between the model proposed by Benth et al. and Bjerksund et al.

In the last modeling section we price a call-option. The calculation is not straight forward, because the delivery period rather than a fixed time in the future, causes problems. In Bjerksund et al. case, we can use Black -76 model, which is a expansion of the Black & Scholes option pricing model, though the underlying is replaced by the swap price. In Benth et al. case we need to approximate the call option price using Monte Carlo simulations.

### 3.1 Swap Pricing

This section specify the swap price functions for each model, ready for modeling purposes. We highlight the impact on the volatility function and hence the swap price, when time to maturity decrease and when delivery time increase. The extreme event will be visual in the swap price. We stress the fact that the direct approach given by Benth et al. needs a smoothing technique for the forward price at time  $t = 0$ , this will not be established here. We use  $f(0, u_i) = 100 \forall i$ , and  $F(0, T_1, T_2) = 100$  as standard.

The following plots and estimates are done with Monte Carlo, 10000 simulations. For each model we have plotted the autocorrelation for the mean swap price and mean log-returns, the purpose is to detect a trend or a repeating pattern. We visualize some swap price paths, to easily observe the impact of the

volatility function. And finally we normalize the log-returns and compare to the normal distribution in a QQ-plot. The time perspective are trading days, hence we divide the volatility function by  $\sqrt{250}$ .

### Bjersund et al. approach

First let us derive the log-returns in order to say what the expected return on the contract is. The log-returns gives us the expected return from one period to another. It is normal to assume the returns are normal distributed, as we will derive under. This explicit expression is only possible to derive with this model. For Benth et al. we need to approximate it numerically.

$$\begin{aligned}
G(t) &= \ln(F(t, T_1, T_2)) - \ln(F(t-1, T_1, T_2)) \\
&= \ln\left(\frac{F(t, T_1, T_2)}{F(t-1, T_1, T_2)}\right) \\
&= \ln\left(\frac{F(t-1, T_1, T_2)e^{-\frac{1}{2}\int_{t-1}^t \Sigma^2(s, T_1, T_2)ds} + \sqrt{\int_{t-1}^t \Sigma^2(s, T_1, T_2)ds} X}{F(t-1, T_1, T_2)}\right) \\
&= -\frac{1}{2}\int_{t-1}^t \Sigma^2(s, T_1, T_2)ds + \sqrt{\int_{t-1}^t \Sigma^2(s, T_1, T_2)ds} X
\end{aligned}$$

And

$$\begin{aligned}
E[G(t)] &= E\left[-\frac{1}{2}\int_{t-1}^t \Sigma^2(s, T_1, T_2)ds + \sqrt{\int_{t-1}^t \Sigma^2(s, T_1, T_2)ds} X\right] \\
&= -\frac{1}{2}\int_{t-1}^t \Sigma^2(s, T_1, T_2)ds
\end{aligned}$$

and

$$\begin{aligned}
Var(G(t)) &= Var\left(-\frac{1}{2}\int_{t-1}^t \Sigma^2(s, T_1, T_2)ds + \sqrt{\int_{t-1}^t \Sigma^2(s, T_1, T_2)ds} X\right) \\
&= \int_{t-1}^t \Sigma^2(s, T_1, T_2)ds
\end{aligned}$$

Hence the log-returns are independent and normal distributed with expectation and variance given above. In addition the log-returns, since the Brownian Motion process is a process with independent and stationary increments where the increments are normally distributed, will become independent and stationary.

Bjersund et al. introduce a specific volatility function at time  $t \leq u$ , where  $u$  is delivery time, given by  $\sigma(t, u) = \left(\frac{a}{u-t+b} + c\right)$ ,  $a, b, c > 0$ . The extreme events are,

$$\begin{aligned}
\lim_{t \rightarrow u} \sigma(t, u) &= \lim_{t \rightarrow u} \left(\frac{a}{u-t+b} + c\right) \\
&= \frac{a}{b} + c
\end{aligned}$$

As time to maturity decrease the volatility function approximate a maturity effect  $\frac{a}{b}$  and an annual volatility average  $c$ .

$$\begin{aligned}\lim_{u \rightarrow \infty} \sigma(t, u) &= \lim_{u \rightarrow \infty} \left( \frac{a}{u - t + b} + c \right) \\ &= c\end{aligned}$$

When time to delivery increase, the annual volatility average is the only decisive factor.

From chapter 2 we remember the approximation to a log-normal swap dynamic *given* a log-normal forward dynamic with known volatility function done by Bjerksund et al.

$$\begin{aligned}F(t, T_1, T_2) &= \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(0, u) du + \int_0^t \sigma(s, T_2) F(s, T_1, T_2) \\ &\quad - \int_{T_1}^{T_2} \frac{\partial \sigma(s, u)}{\partial u} \frac{u - T_1}{T_2 - T_1} F(s, T_1, u) du dW(s) \\ &\approx \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(0, u) du + \int_0^t \sigma(s, T_2) F(s, T_1, T_2) dW(s)\end{aligned}$$

And dynamic given by

$$dF(t, T_1, T_2) = F(t, T_1, T_2) \Sigma(s, T_1, T_2) dW(t) \quad (3.1)$$

Here

$$\Sigma(t, T_1, T_2) = \left( \frac{a}{T_2 - T_1} \ln \left( \frac{T_2 - t + b}{T_1 - t + b} \right) + c \right)$$

Using Ito's formula on (3.1), we get the following swap price at time  $t$ , with delivery period  $[T_1, T_2]$

$$F(t, T_1, T_2) = F(t - 1, T_1, T_2) e^{-\frac{1}{2} \int_{t-1}^t \Sigma^2(s, T_1, T_2) ds + \sqrt{\int_{t-1}^t \Sigma^2(s, T_1, T_2) ds} X} \quad (3.2)$$

where

$$X \sim N(0, 1)$$

In order to derive the swap price, we need to calculate the volatility function  $\int_t^T \Sigma^2(s, T_1, T_2) ds$ . The calculation follows as in [9]

$$\begin{aligned}\int_t^T \Sigma^2(s, T_1, T_2) ds &= \frac{a^2}{(T_2 - T_1)^2} \int_t^T \left( \ln \left( \frac{T_2 - s + b}{T_1 - s + b} \right) \right)^2 ds \\ &\quad + \frac{2ac}{T_2 - T_1} \int_t^T \ln \left( \frac{T_2 - s + b}{T_1 - s + b} \right) ds \\ &\quad + c^2(T - t)\end{aligned}$$

The first integral is given by,

$$\int_t^T \left( \ln \left( \frac{T_2 - s + b}{T_1 - s + b} \right) \right)^2 ds = u(T) - u(t)$$

where, for  $s \leq T_1$ ,

$$\begin{aligned} u(s) = & (T_2 + b - s)(\ln(T_2 + b - s))^2 \\ & - 2(T_2 + b - s) \ln(T_2 + b - s) \ln(T_1 + b - s) \\ & + 4a \ln(T_2 - T_1) \ln \left( \frac{T_1 + b - s}{T_2 - T_1} \right) \\ & - 2(T_2 - T_1) \text{dilog} \left( \frac{T_2 + b - s}{T_2 - T_1} \right) \\ & + (T_2 + b - s)(\ln(T_1 + b - s))^2 - 2(T_2 - T_1) \end{aligned}$$

where the dilogarithm is defined for  $x \geq 0$  as,

$$\text{dilog}(x) = - \int_1^x \frac{\ln(s)}{s-1} ds$$

Or in our context we may approx it numerically as,

$$\text{dilog}(x) \approx \begin{cases} \sum_{k=1}^n \frac{(x-1)^k}{k^2}, & x \in [0, 1] \\ -\frac{1}{2}(\ln(x))^2 - \sum_{k=1}^n \frac{((1/x)-1)^k}{k^2}, & x > 1 \end{cases}$$

for n large. The last integral is given by,

$$\begin{aligned} \int_t^T \ln \left( \frac{T_2 - s + b}{T_1 - s + b} \right) ds = & (T_2 + b - T) \ln(T_2 + b - T) \\ & - (T_1 + b - T) \ln(T_1 + b - T) \\ & - (T_2 + b - t) \ln(T_2 + b - t) \\ & + (T_1 + b - t) \ln(T_1 + b - t) \end{aligned}$$

The calculation procedure to the volatility function is rather complex, compared to the one we will derived in Benth et al. case.

The swap price can no be simulated following this algorithm

- Set initial value of the swap price.
- Set  $T_1$  and  $T_2$  to the desired time period of the contract.
- For each time  $t \leq T_1$  derive the volatility function  $\Sigma^2(t, T_1, T_2)$  and the respectively swap price  $F(t, T_1, T_2)$  (3.2).
- Repeat the procedure in order to be able to estimate expectation and standard error.

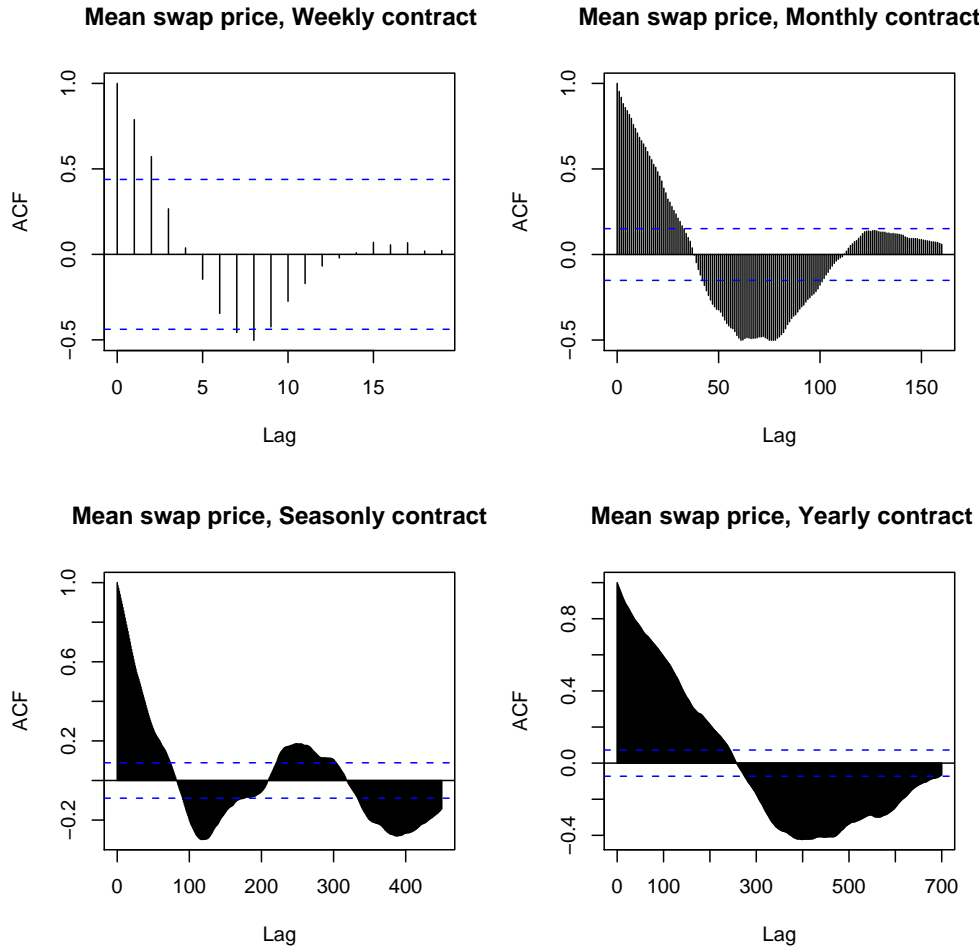


Figure 3.1: Autocorrelation of the swap contract, given by Bjerksund et al.

In figure 3.1 we have plotted for each contract the autocorrelation of the swap price. It shows a slowly decreasing trend indicating strong positive correlation and no-stationarity. But as time from origin increase, the dependencies decrease. In addition the pattern repeat itself though with weaker dependencies.

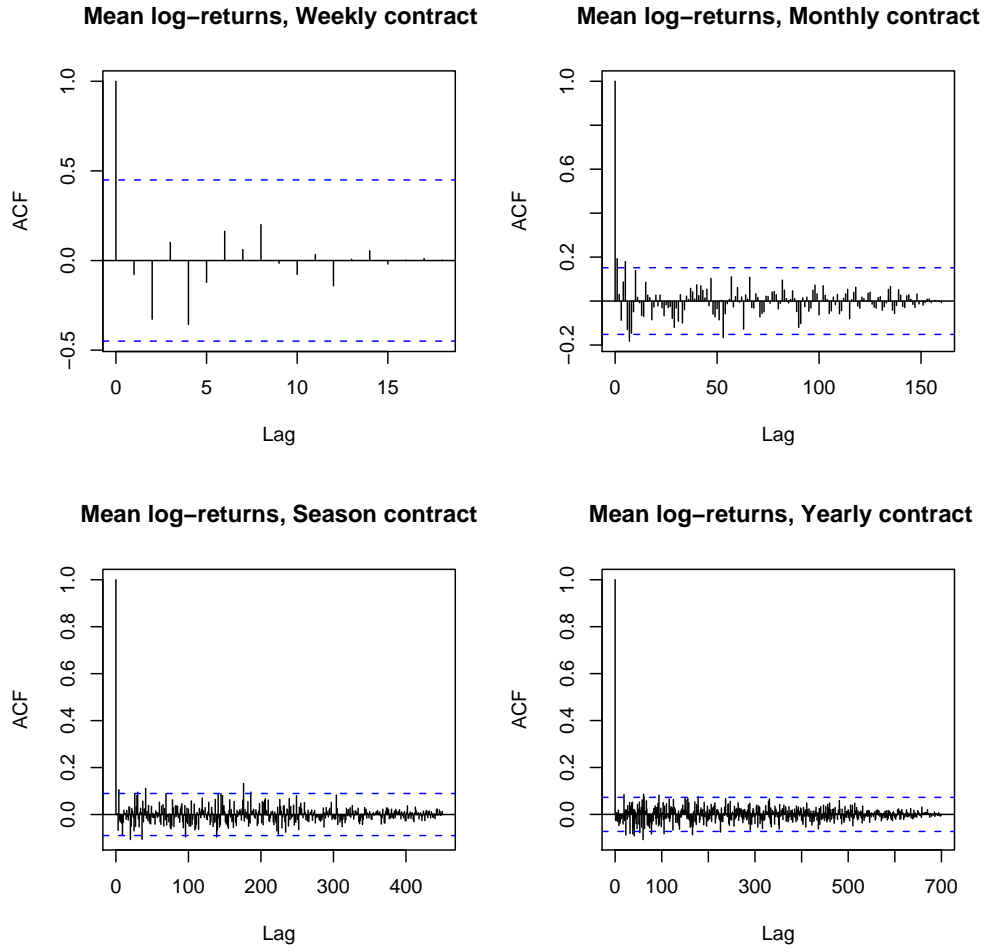


Figure 3.2: Autocorrelation of the log returns, given by Bjerksund et al.

In figure 3.2 the autocorrelation of the log returns, for each contract have been plotted. The autocorrelation clearly show that they are independent and identical distributed. For each contract almost all log-returns fall under the 5 % confidence interval. Showing our calculation over is justified. The log-returns become independent and normal distributed. This justifies that the log returns are stationary.



TD	$T_1 \rightarrow T_2$	sd(F)	sd(G)	skew(G)	kurt(G)	DF-test(F)	DF-test(G)
25	7	0.7271	0.0075	-0.1535	-0.930	0.6609	0.6312
168	28	2.4838	0.011	0.2109	0.1539	0.963	0.01
480	120	9.28	0.0130	0.0976	-0.094	0.4099	0.01
730	365	9.7	0.0131	0.1559	0.013	0.4452	0.01

Table 3.1: Simulated values for each contract under Bjerksund et al. model. sd(F) and sd(G)- annual standard deviation to the swap price and log-returns, skew(G)-skewness of the log-returns, kurt(G)-kurtosis of the log-returns, DF-test- Dickey Fuller test for the swap price and log returns

In table 3.1 estimated value for the standard deviation of the swap price and log-return are estimated for each contract. In addition we have the skewness and kurtosis for the log-return. The standard deviation for the swap price increase as time of delivery of the contract increase. This deviates from empirical data, e.g. [6]. But the annual standard deviation of the log-returns decrease as contract time increase, indicating a less risky investment / the probability of profit decrease as contract time increase. The skewness of the log-return clearly indicating a symmetric distribution, but the kurtosis seem to differ from the normal hypothesis for contracts with short delivery period. In addition we test for stationarity using the Dickey-Fuller test. The Dickey-Fuller test provides us with a p-value. The lower p-value the better our null hypothesis i.e. the log-returns become stationary is justified. The above estimations justifies previous calculations. The swap price will not become stationary, but the log returns do. Though the estimated p-value for the weekly contract, have a high p-value. This might be due to the short delivery period.

### Benth et. al approach

The important structure for modeling forward dynamic and hence finally the swap price, is the structure of the volatility function  $\sigma(t, u)$ . In Bjerksund et al. case the volatility function was given, in this case we will look at three different volatility functions represented in [6], each trying to capture the stylish feature of the forward curve volatility.

1. Schwartz (1997) have a exponential decay volatility function given by :

$$\sigma(t, u) = ae^{-b(u-t)}, \quad a, b > 0$$

- a - Represents average annual volatility over contracts representative in the market, hence day, week, month, season and year. If  $a = 0.56$ , the annual volatility is 56%.
- b - Controls the maturity effect, the closer  $t$  is to  $u$ , the closer we are to maturity of the contract, the more b affects the volatility.

The extreme events are,

$$\begin{aligned}\lim_{t \rightarrow u} \sigma(t, u) &= \lim_{t \rightarrow u} a e^{-b(u-t)} \\ &= a\end{aligned}$$

That is, as time to maturity approaching, the value of  $b$  will affect the volatility, the higher value of  $b$  the sharper the volatility is close to maturity. But at time of maturity the only decisive factor is the annual volatility, and the volatility function catches not the seasonal structure.

$$\begin{aligned}\lim_{u \rightarrow \infty} \sigma(t, u) &= \lim_{u \rightarrow \infty} a e^{-b(u-t)} \\ &= 0\end{aligned}$$

That is, when we are far from delivery time, the swap price will hardly be affected by the volatility.

2. The next volatility function only catch the seasonal part, given as a Fourier series

$$\sigma(t, u) = c + d \sin\left(\frac{2\pi t}{365}\right) - f \cos\left(\frac{2\pi t}{365}\right), \quad c, d, f \text{ constants}$$

The volatility function will have a repeating pattern.

3. Benth and Koekebakker (2008) introduced a volatility function, with a clear separation of maturity and season effect.

$$\sigma(t, u) = a e^{-b(u-t)} + c + d \sin\left(\frac{2\pi t}{365}\right) - f \cos\left(\frac{2\pi t}{365}\right), \quad a, b, c, d, f \text{ constants}$$

Here they combine the maturity model proposed by Schwartz and a Fourier series. As for the other two volatility functions we look at the extreme events:

$$\begin{aligned}\lim_{t \rightarrow u} \sigma(t, u) &= \lim_{t \rightarrow u} a e^{-b(u-t)} + c + d \sin\left(\frac{2\pi t}{365}\right) - f \cos\left(\frac{2\pi t}{365}\right) \\ &= a + c + d \sin\left(\frac{2\pi t}{365}\right) - f \cos\left(\frac{2\pi t}{365}\right)\end{aligned}$$

The closer we get to maturity, the volatility collapse to average volatility and season effect, but as for the Schwartz model, close to maturity, the value of  $b$ , will play a great importance.

$$\begin{aligned}\lim_{u \rightarrow \infty} \sigma(t, u) &= \lim_{u \rightarrow \infty} a e^{-b(u-t)} + a + d \sin\left(\frac{2\pi t}{365}\right) - f \cos\left(\frac{2\pi t}{365}\right) \\ &= c + d \sin\left(\frac{2\pi t}{365}\right) - f \cos\left(\frac{2\pi t}{365}\right)\end{aligned}$$

On the other extreme, when time to delivery increase the volatility will only be an additive of average annual volatility and season effect.

These three volatility functions each represents important features considering the swap price. Before we start modeling and see which may be the best suited for the job, a good guess will be volatility function number three. Where both time to maturity and seasonality effect are represented.

The constants  $a, b, c, d, f, g$  are being estimated from maximum-likelihood method, with data of swap contracts traded at Nord Pool in the period 1996-2004. The estimation procedure are explained in [6].

Given the volatility function  $\sigma(t, u)$ , we need to estimate the swap price. We will approximate it under the no-arbitrage condition (2.9) in chapter 2. That is we approximate the swap price given log-normal forwards, and not the "direct approach". Remember the  $u$ 's are delivery *points* in the delivery period  $[T_1, T_2]$ .

$$\begin{aligned}
F(t, T_1, T_2) &= \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(t, u) du \\
&\approx \sum_{i=1}^n \frac{1}{T_2 - T_1} f(t, u_i) \Delta_i \\
&= \frac{1}{T_2 - T_1} \sum_{i=1}^n f(t, u_i) (u_{i+1} - u_i) \\
&= \frac{1}{T_2 - T_1} \sum_{i=1}^n f(t-1, u_i) e^{-\frac{1}{2} \int_{t-1}^t \sigma^2(s, u_i) ds + \int_{t-1}^t \sigma(s, u_i) dW(s)} \\
&= \frac{1}{T_2 - T_1} \sum_{i=1}^n f(t-1, u_i) e^{-\frac{1}{2} \int_{t-1}^t \sigma^2(s, u_i) ds + \sqrt{\int_{t-1}^t \sigma^2(s, u_i) ds} X_i}
\end{aligned} \tag{3.3}$$

The second approximation is a regular Riemann approximation. We let

$$\begin{aligned}
T_1 &= u_1 < u_2 < \dots < u_{n-1} < u_n = T_2 \\
\Delta_i &= u_{i+1} - u_i = 1 \text{ day} \\
X_i &= \frac{\int_{t-1}^t \sigma(s, u) dW(s)}{\sqrt{\int_{t-1}^t \sigma^2(s, u) ds}}
\end{aligned}$$

which become a standard normal distributed. In order to have a correlation between delivery points  $u_i$  and  $u_j$ ,  $i \neq j$ , for volatility functions which are not independent i.e  $\sigma(t, u) \neq \sigma(t)\sigma(u)$  the correlation between  $X_i$  and  $X_j$  is given by:

$$\begin{aligned}
corr(X_i, X_j) &= \frac{cov(X_i, X_j)}{sd(X_i)sd(X_j)} \\
&= \frac{E[X_i X_j] - E[X_i]E[X_j]}{sd(X_i)sd(X_j)} \\
&= \frac{\int_{t-1}^t \sigma(s, u_i) \sigma(s, u_j) ds}{\sqrt{\int_{t-1}^t \sigma^2(s, u_i) ds} \sqrt{\int_{t-1}^t \sigma^2(s, u_j) ds}}
\end{aligned} \tag{3.4}$$

The common way to develop correlation matrix is to use a Cholesky decomposition. In our setting, with only one Brownian Motion, the correlation between two delivery points are perfectly correlated, independent of the volatility model. This means for each delivery point  $u$  at time  $t$  of the forwards, the same BM will be applied. In addition the log-returns will become stationary.

The simulation follows the same path as for Bjerksund et al. but we need to sum the forward prices for each time  $t \leq T_1$ , in order to get the swap price at time  $t$ .

Our main object of interest will be volatility function three, the following plots are in the purpose of justify the desired volatility function, and give a perspective.

1. Simulated values for volatility function one,  $\sigma(t, u) = ae^{-b(u-t)}$ ,  $a = 0.68$ ,  $b = 0.784$

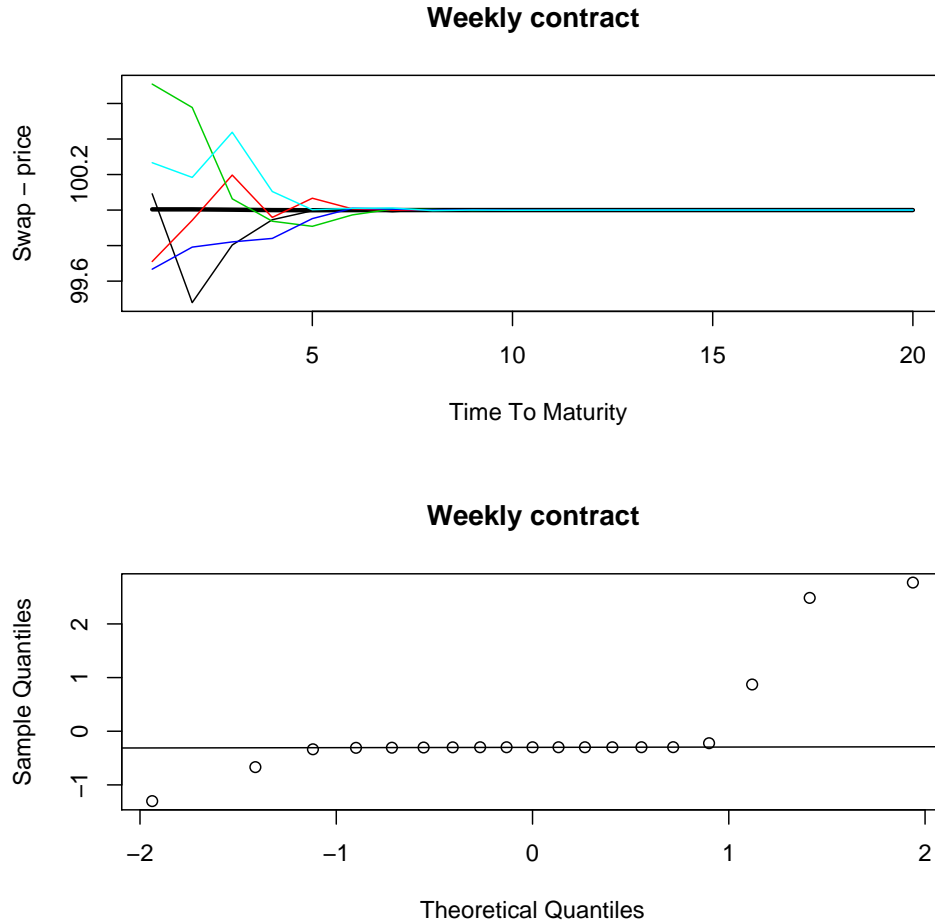


Figure 3.3: Weekly contract, volatility model 1, given by Benth et al.

The important feature in figure 3.3 is easily observed, this volatility function will only give a impact on the swap price, close to maturity. Further the hypothesis of normal distributed log-returns fails. The QQ-plot indicates heavy right tail.

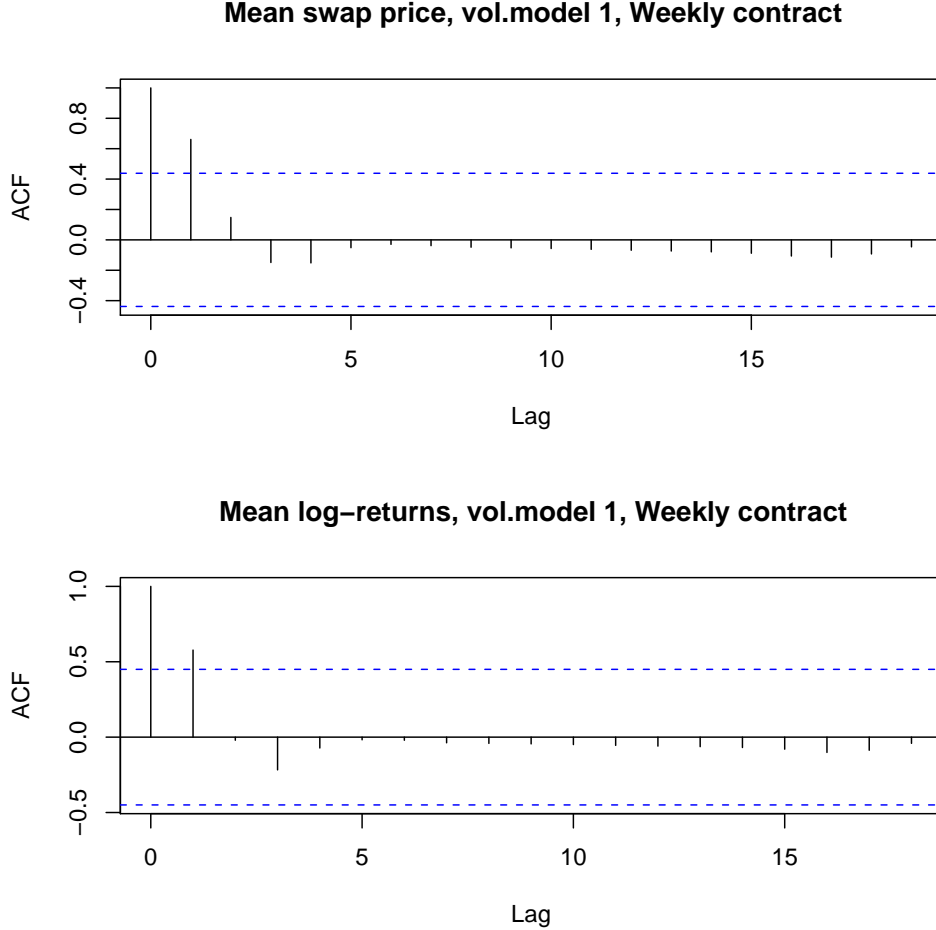


Figure 3.4: Weekly contract, volatility model 1, given by Benth et al.

In figure 3.4 autocorrelation for both mean swap price and mean log-returns are given. The plots clearly indicate stationary and independent log-returns. Second we do not observe a repeating pattern, due to stationarity of the volatility function.

2. Simulated values for volatility function two,  $\sigma(t, u) = c + d \sin(\frac{2\pi t}{365}) - f \cos(\frac{2\pi t}{365})$ ,  $c = 0.190$ ,  $b = 2.667$ ,  $d = 0.066$ ,  $f = -0.179$

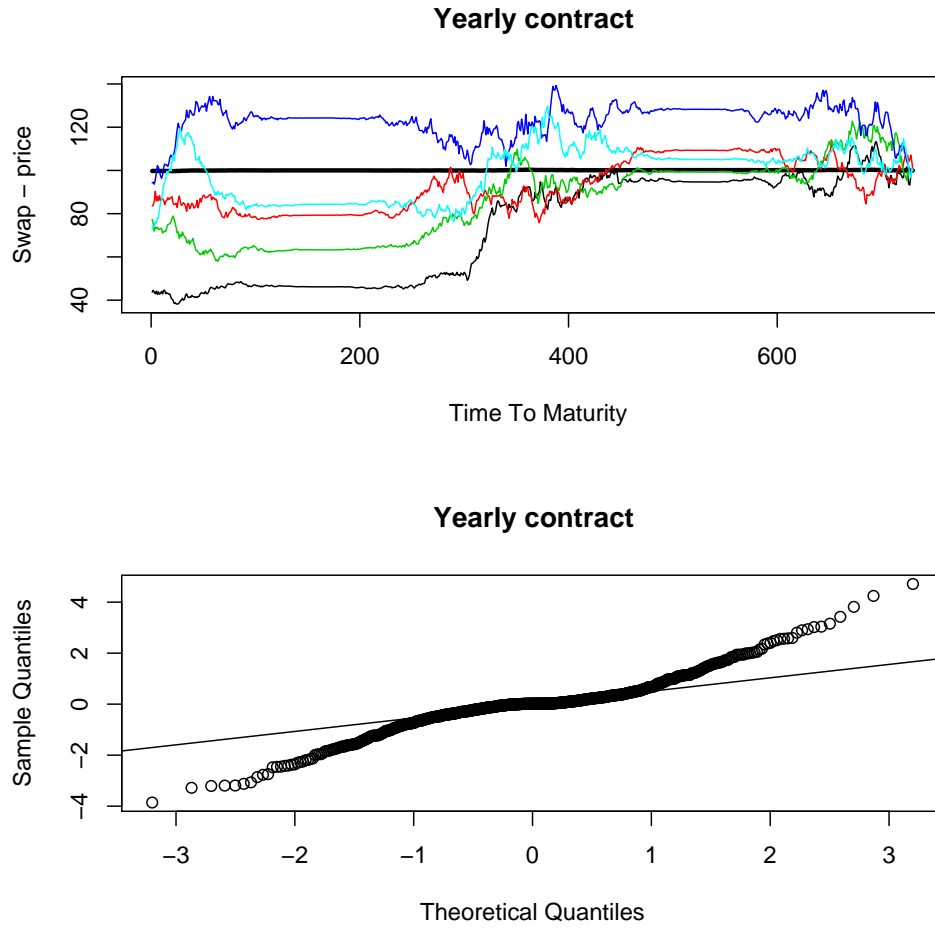


Figure 3.5: Yearly contract, volatility model 2, given by Benth et al.

Easily observed in figure 3.5 is the desired seasonal behavior of the swap price. Within a period of 250 days, the pattern repeats itself. The maturity effect, which was visible in volatility function one, are not included here. This is visible in the end of delivery. Second, we see the approximation to the normal distribution is bad. The plot clearly indicates both a left and right heavy tail.

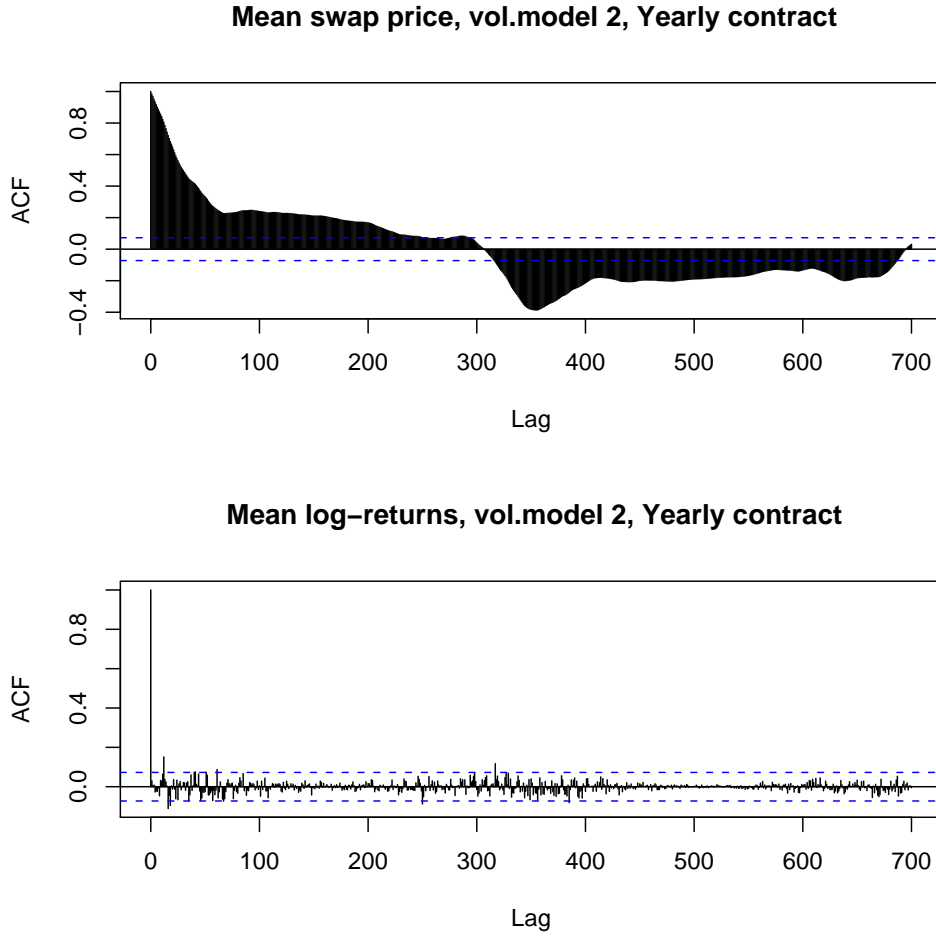


Figure 3.6: Yearly contract, volatility model 2, given by Benth et al.

The autocorrelation function for the mean swap price in figure 3.6 clearly show strong dependencies indicating a no-stationary series. The log-returns slightly reveal a repeating patten due to the seasonality. In addition only five lags fall below the confidence interval, indicating the log-returns are iid. The stationarity effect have almost being wiped out. It is visual, but highly modified.

3. Simulated values for volatility function three,  $\sigma(t, u) = ae^{-b(u-t)} + c + d\sin(\frac{2\pi t}{365}) - f\cos(\frac{2\pi t}{365})$ ,  $a = 0.604$ ,  $b = 2.848$ ,  $c = 0.161$ ,  $d = 0.018$ ,  $f = -0.065$

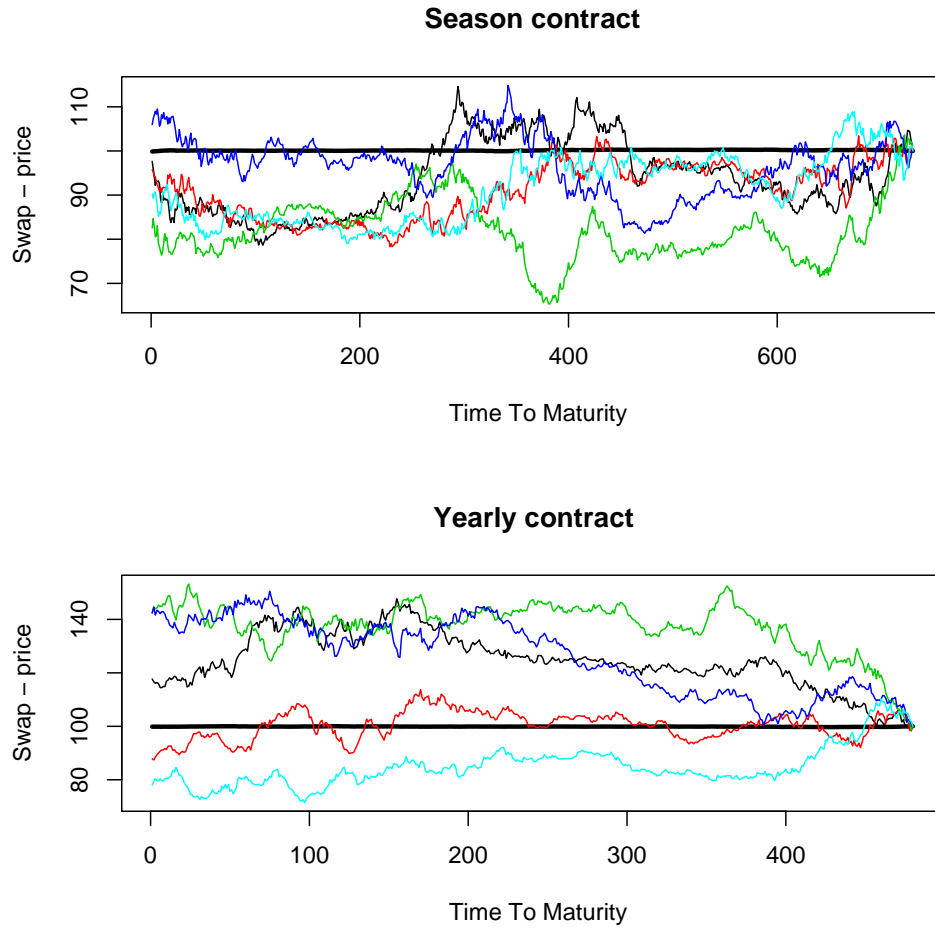


Figure 3.7: Swap price paths, Benth et al.

In figure 3.7 we have showed path simulations for a seasonal and a yearly contract. The season fluctuations are clearly visual. And in addition we see the maturity effect. But since we have not included jumps, the spike effect are not visual. We see the price span of the seasonal contract is  $[70, 110]$  but for the yearly contract is  $[80, 150]$ . The increase in the standard deviation for longer dated contracts are estimated in table 3.2.



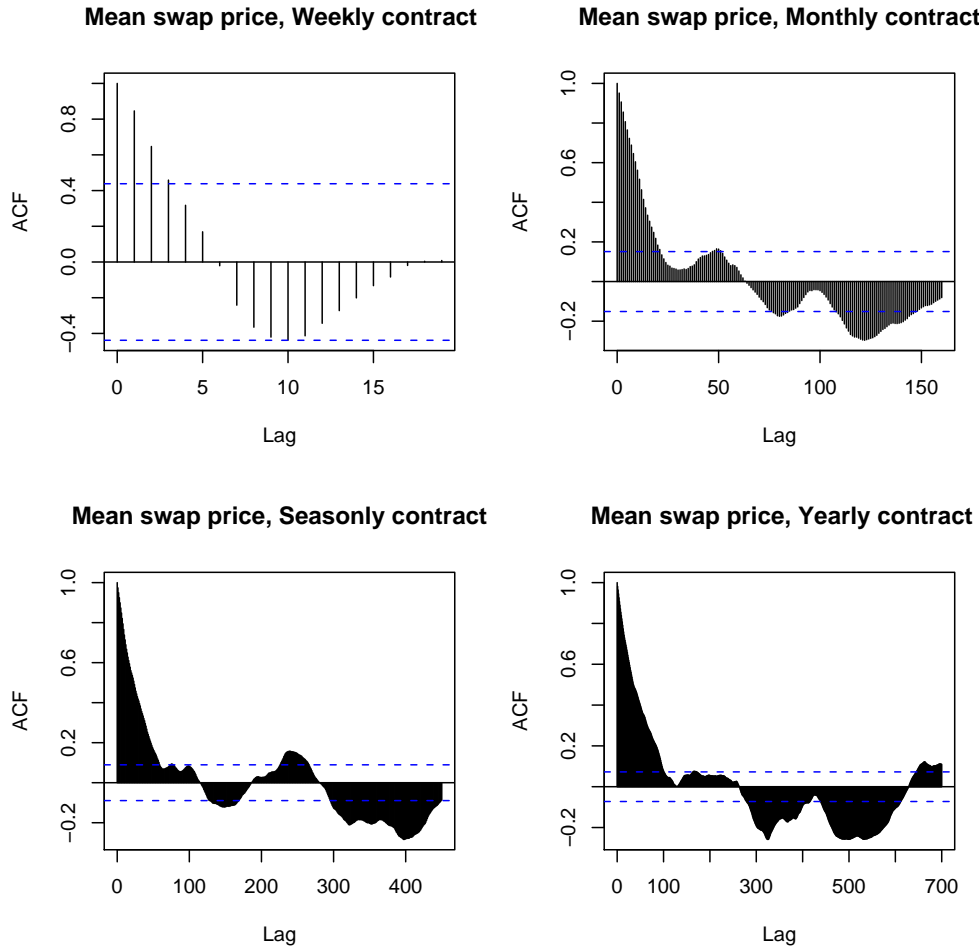


Figure 3.8: Autocorrelation of the swap contract, given by Benth et al.

In figure 3.8 we see the same behavior as in the model proposed by Bjerk-sund et al. The swap price are highly dependent on previous time, as constructed when simulating the prices, this also justifies the no-stationarity. In addition we easily observe a repeating pattern this is due to the season effect given through the volatility function.

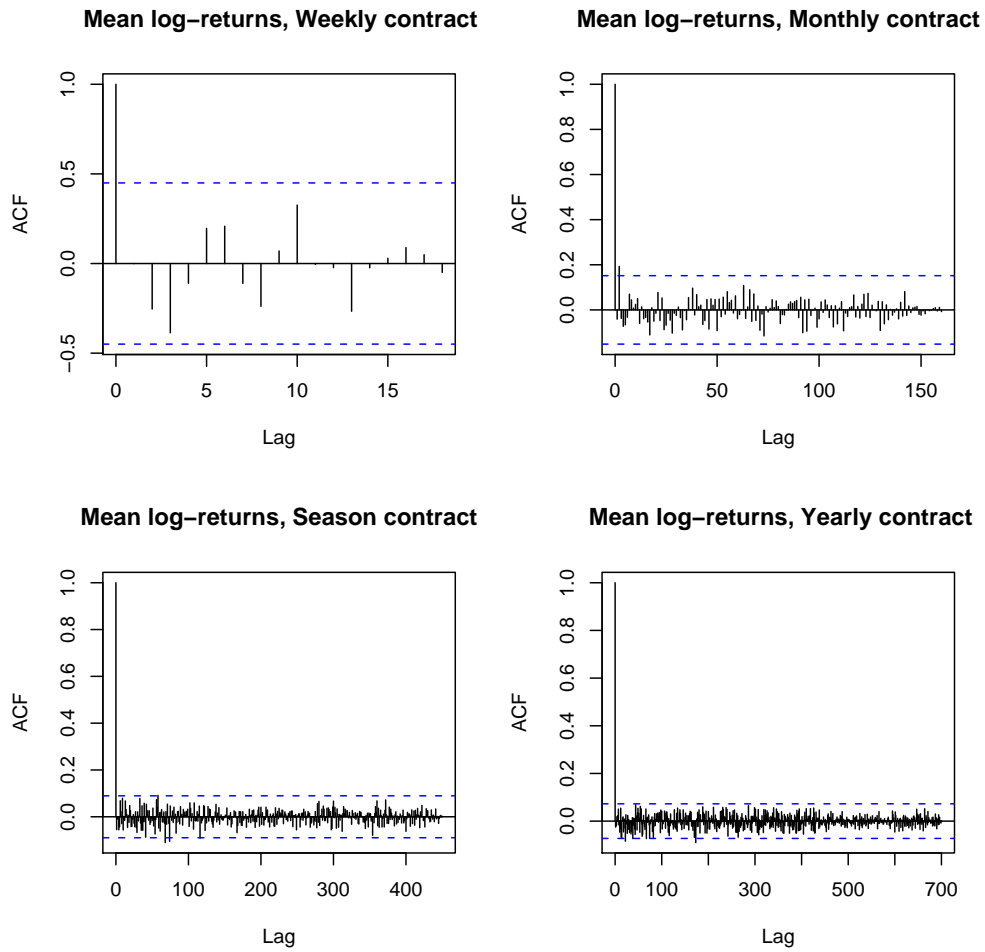


Figure 3.9: Autocorrelation of the log returns, given by Benth et al.

The autocorrelation of the log returns in figure 3.9 clearly indicate that they are independent and identical distributed. In addition we observe the log-returns change over the lags. Though not as clear as in figure 3.6 we see a trace of a repeating pattern, this due to the seasonality. The no-stationarity is clearly visual but since the autocorrelation is under the confidence interval the log-returns are nothing but noise, as expected.

TD	$T_1 \rightarrow T_2$	sd(F)	sd(G)	skew(G)	kurt(G)	DF-test(F)	DF-test(G)
25	7	0.5123	0.0024	0.1528	-0.7627	0.5003	0.2161
168	28	0.7522	0.0018	-0.080	0.7646	0.9553	0.01
480	120	3.2992	0.0017	-0.1343	1.3783	0.9734	0.01
730	365	0.9237	0.0017	-0.3945	0.402	0.629	0.01

Table 3.2: Simulated values for each contract under Benth et al. model. sd(F) and sd(G)- annual standard deviation to the swap price and log-returns, skew(G)-skewness of the log-returns, kurt(G)-kurtosis of the log-returns,DF-test- Dickey Fuller test for the swap price and log returns

Plots of the simulated swap price will be presented in next section, in comparison with Bjerksund et al. model. In table 3.2 we observe the annual standard deviation of the swap price slightly increase as time of delivery increase. Except for the seasonal contract, where the annual standard deviation deviate significant from the other contracts. The explanation to this, might be due to a regular season effect in the end of the trading time, rises the standard deviation. The annual standard deviation of the approximated log-returns, are decreasing as time of delivery period increase. Which is in comparison to empirical analysis done in [6]. The skewness and kurtosis of the log-returns show a good approximation to the normal distribution. The p-value given by the Dickey-Fuller test justifies the observed behavior in figure 3.8 and 3.9. The swap price is not stationary but the log-returns become stationary at least for contacts with longer delivery period than a week.

### Comparison of Bjerksund et al. and Benth et al.

Now we will investigate differences, if any, in the two models. Bjerksund et al. model had a specific volatility function, depending on time to maturity. In Benth et al. case we looked at three different volatility models, but the volatility model including both season and maturity effect are the one closest to approximate reality, therefore the swap price with volatility function three will be our comparison to Bjerksund et al.

Our main focus is whether the model for the swap price by Benth et al. resembles a log-normal model, hence our comparison will be the swap price approximation to a log-normal variable done by Bjerksund et al.

Further we investigate the difference between Bjerksund et al. and Benth et al. approach. Since no approximation is done with the Benth et al. model, this will be the "real" price, and our focus will be, how well the model by Bjerksund et al. is to the "real" price.

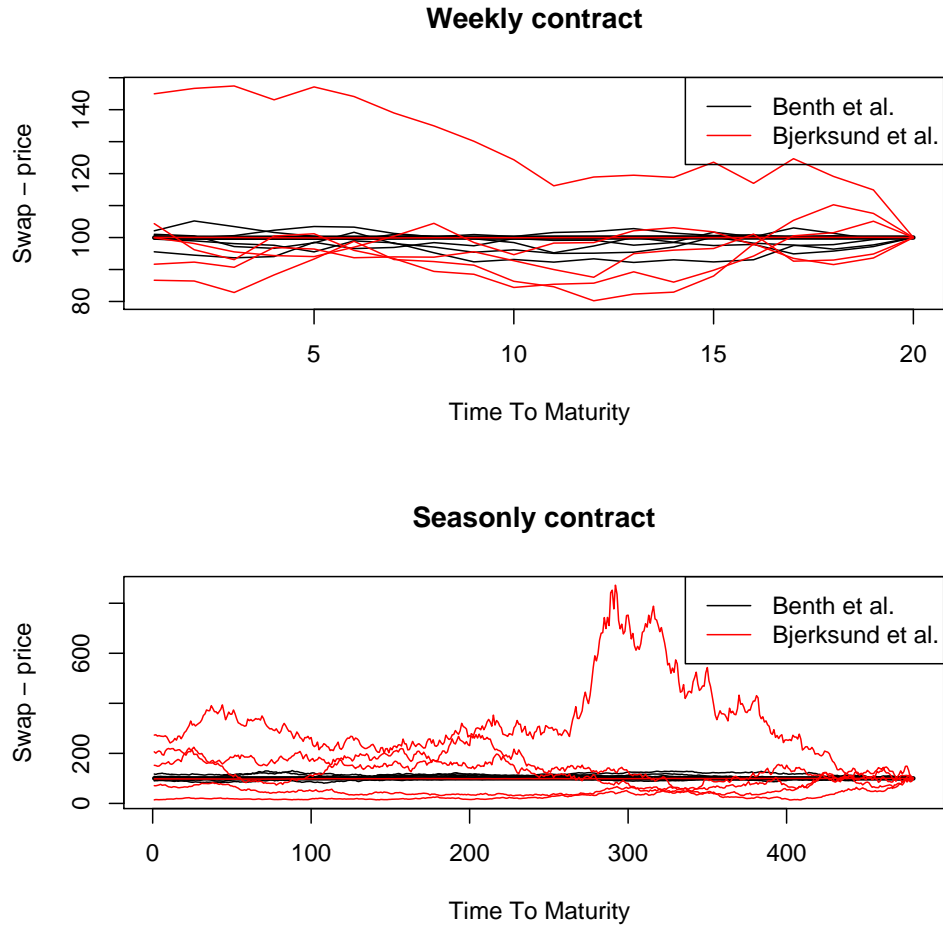


Figure 3.10: Swap-prices, weekly and seasonal contract

In figure 3.10 we have plotted swap price paths for a weekly and seasonal contract. We observe a higher volatility in the price path estimated with Bjerksund et al. than the price paths simulated by Benth et al. This might be due to different volatility functions. Or the log-normal approximation by Bjerksund et al. better capture the extreme events that might happen, compared to the sum of log-normal variables given by Benth et al. As a natural consequence to this, the variability in the swap price from Bjerksund et al. will be greater, then to Benth et al. This is also observable in table 3.1 and table 3.2.

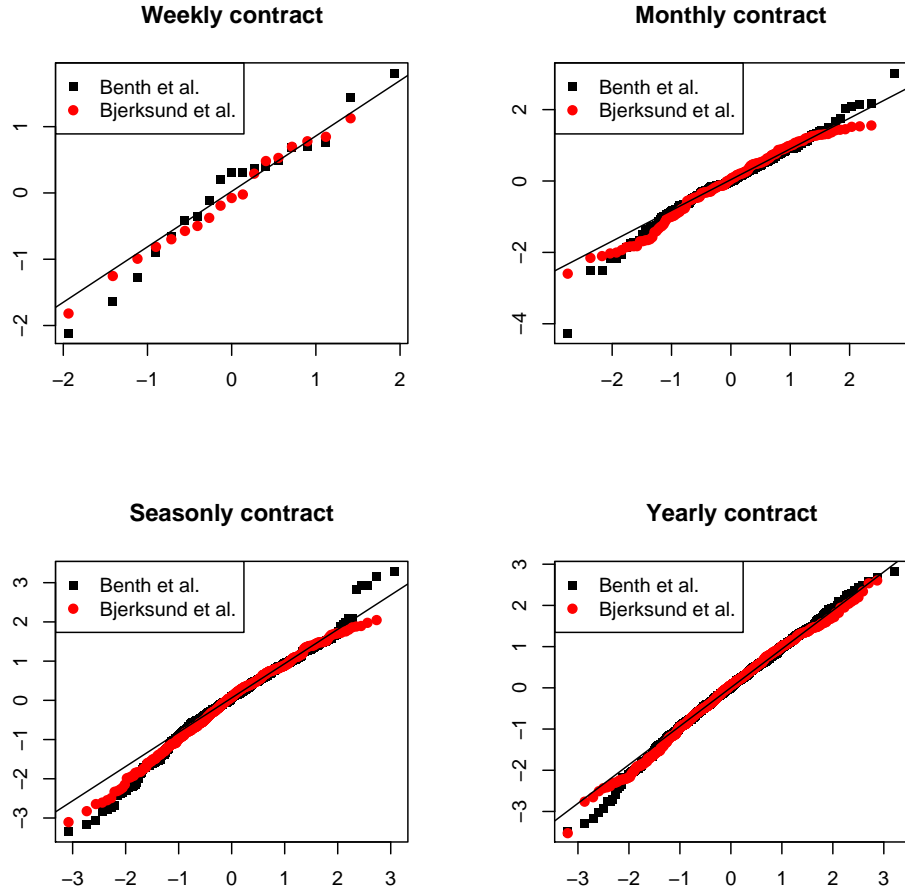


Figure 3.11: Normalized log-returns

In figure 3.11, we compare normalized log returns for each model and with different delivery period. The main feature, as time of delivery period increase is a better fit to the standard normal distribution. Bjerksund et al. approach the normal distribution faster than Benth et al., which directly can be transferred to the nature of the given swap model. The important observation on the other hand is the good approximation to a standard normal distribution given the model proposed by Benth et al. But in addition we see a more heavy left tail for the yearly contract.

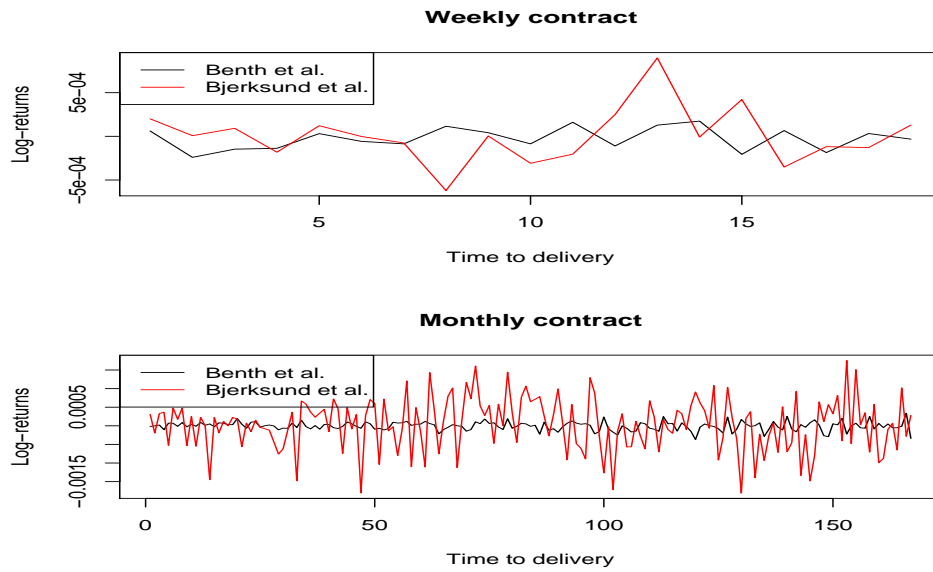


Figure 3.12: Log-returns

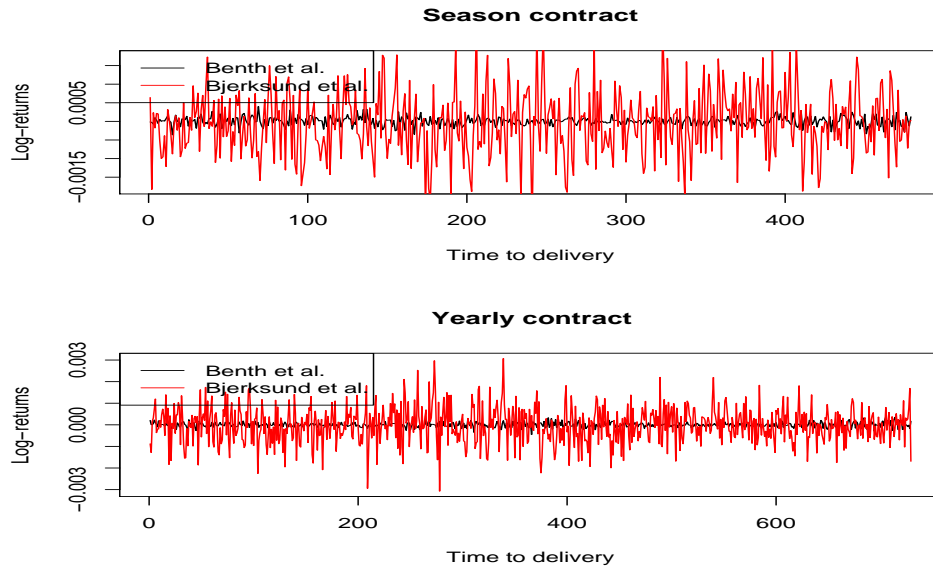


Figure 3.13: Log-returns

In figure 3.12 and 3.13 we have plotted the estimated log-returns for each contract. As we have paid attention to in table 3.1 and table 3.2, the standard deviation in the log-returns given by Bjerksund et al. was greater than the standard deviation given by Benth et al. In addition in figure 3.10 the swap price path given by Bjerksund have a greater volatility than the swap price paths

given by Benth et al. This is truly visual in 3.12 and 3.13. The log-returns from Bjerksund et al. have greater volatility than the 'real' log-returns given by Benth et al.

## Summary

The previous section showed the importance of a good volatility function.

We started out with two different models. The model proposed by Bjerksund et al. showed strong correlation and a significant trend when we looked at the autocorrelation for the swap price. But when we turned to the autocorrelation of the log-returns showed us independences and stationary log-returns.

With the model proposed by Benth et al. we started out with three different volatility models. With volatility one, the QQ-plot showed a deviation to the normal distribution. The autocorrelation of the mean swap price and log returns became stationary. The trend, for time points close to maturity was being wiped out.

For volatility model two the QQ-plot showed a deviation to the normal distribution, the tail were to heavy. Autocorrelation plots showed both a trend and a seasonal effect. The log-returns had become independent and stationarity. It was slightly visual a change in the log-returns due to the seasonality.

Finally with the third volatility model, we observed a repeating pattern and a increase in the volatility close to maturity. The autocorrelation for the mean swap price showed dependencies from previous time-lags. But the repeating pattern in the mean swap price, was hardly visual in the log-returns. And the log-returns became independent and identical distributed.

When we looked at the price paths for each model given by Bjerksund et al. and Benth et al. the differed significantly. In addition we saw the significant difference in the standard deviation, indicating a significant difference in the swap price. The log-returns in addition would give different returns. The difference swap price paths and hence the log-returns can be due to the different volatility functions. For each each model the QQ-plot justified the normal hypothesis.

In other words, for different volatility models, we generate a price path more similar to what we see in empirical data. We could detect a pattern in the log-returns for volatility model 2 and 3. The different swap price paths and hence the log-returns of each model differed significant.

## 3.2 Option Pricing

This section derives the call option price written on a swap for both of the models, modeled above. The structure of the call option price will be very similar to the one we are used to when the underlying derivative is a single

geometrical Brownian motion, but in our context the underlying is a swap contract model as a log-normal forward price, as in Bjerksund et al. then we need to use the Black -76 formula instead. In Benth et. al. case we need to approximate the price by Monte Carlo simulations. In addition we will derive the delta-hedge, for both of the models.

## Black -76 Model

The Black model was first introduced in 1976 by Fischer Black. The reason for a new model, contrast to the Black & Scholes (1973), was the dynamic of commodities which had non-randomness structure e.g. commodities depending on season. The season dynamic will not be reflected in the Black & Scholes model, because here the underlying derivative is a geometric Brownian motion,  $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$ , where  $\mu$  and  $\sigma$  are constants and  $W(t)$  is a Brownian motion. This structure will not catch the season effect or e.g. a increase in price prior maturity with a fall in price following the harvest. Therefore Black proposed to model commodities with non-randomness as forwards. Hence the Black -76 model is similar to Black & Scholes but the underlying derivative is a forward instead of geometric Brownian motion. The following theorem stated under, will be the main tool in pricing the swap contracts of interest, but remark this theorem is for a log-normal forward dynamic, the proof is stated in Appendix A.

**Theorem 2** (Black -76 price of a call option). *The price of a call option at time  $t \leq T$  written on a forward contract with delivery time  $\tau$ , where the option has exercise time  $T \leq \tau$ , strike price  $K$  and interest rate  $r$ , is given by*

$$P(t, T, K, T_1, T_2) = e^{-r(T-t)}(f(t, \tau)\Phi(d_1) - K\Phi(d_2)) \quad (3.5)$$

Here

$$d_1 = \frac{\ln(f(t, \tau)/K) + 0.5 \int_t^T \sigma^2(s, \tau) ds}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}}$$

$$d_2 = \frac{\ln(f(t, \tau)/K) - 0.5 \int_t^T \sigma^2(s, \tau) ds}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}}$$

and  $\Phi$  is the cumulative standard normal probability distribution function.

The connection to find the price of a call option with a log-normal swap dynamic as the underlying, is easy to establish, the proof is very similar to the one stated where the underlying were a forward and therefore not included.

**Theorem 3** (Price of a call option). *The price of a call option at time  $t \leq T$  written on a swap contract with delivery period  $[T_1, T_2]$ , where the option has exercise time  $T \leq T_1$ , strike price  $K$  and interest rate  $r$ , is given by*

$$P(t, T, K, T_1, T_2) = e^{-r(T-t)}(F(t, T_1, T_2)\Phi(d_1) - K\Phi(d_2)) \quad (3.6)$$



Here

$$d_1 = \frac{\ln(F(t, T_1, T_2)/K) + 0.5 \int_t^T \Sigma^2(s, T_1, T_2) ds}{\sqrt{\int_t^T \Sigma^2(s, T_1, T_2) ds}}$$

$$d_2 = \frac{\ln(F(t, T_1, T_2)/K) - 0.5 \int_t^T \Sigma^2(s, T_1, T_2) ds}{\sqrt{\int_t^T \Sigma^2(s, T_1, T_2) ds}}$$

and  $\Phi$  is the cumulative standard normal probability distribution function.

Further we establish the delta hedge.

**Proposition 2** (Delta hedge). *The delta hedge of a call option at time  $t \leq T$  written on a swap contract with delivery period  $[T_1, T_2]$ , where the option has exercise time  $T \leq T_1$ , strike price  $K$  and interest rate  $r$ , is given by*

$$\Delta(t; T, K, T_1, T_2) = e^{-r(T-t)} \Phi(d_1) \quad (3.7)$$

Where  $\Phi$  is the cumulative standard normal probability distribution function and  $d_1$  are defined in Theorem 3

### Bjersund et al. approach

We are now able to model the price of a call option where the underlying is a log-normal swap contract, the swap price is defined in section 3.1. The calculation is straight forward.

### Benth et. al approach

As in section 3.1, we have the following model for the swap price

$$F(t, T_1, T_2) = \int_{T_1}^{T_2} \frac{1}{T_2 - T_1} f(t, u) du \quad (3.8)$$

$$\approx \frac{1}{T_2 - T_1} \sum_{i=1}^n f(t, u_i) \quad (3.9)$$

where  $f(t, u)$  is a log-normal process. Hence  $F(t, T_1, T_2)$  is a sum of log-normal variables which is *not* a log-normal process. The approximation procedure will be to do Monte Carlo simulations as explained under.

$$P(t, T, K, T_1, T_2) = e^{-r(T-t)} E_Q[(F(T, T_1, T_2) - K)^+ | \mathcal{F}_t]$$

$$\approx e^{-r(T-t)} \frac{1}{N} \sum_{j=1}^N \left( \sum_{i=1}^n f(t, u_i) e^{-\frac{1}{2} \int_t^T \sigma^2(s, u_i) ds + \sqrt{\int_t^T \sigma^2(s, u_i) ds} X_i} - K \right)^+$$

Where  $N$  is the number of Monte Carlo simulations.

Though the delta hedge is not that clear to calculate, following the approach given in [6], we have following delta hedge.

$$\Delta(t; T, K, T_1, T_2) \approx e^{-r(T-t)} \sum_{i=1}^n \frac{\partial P(t, T, K, T_1, T_2)}{\partial f(t, u_i)} \frac{1}{T_2 - T_1}$$

### Comparison of Bjerk Sund et al. and Benth et al.

Instead of modeling the hole price path, we look at the price at time  $t = 0$ . At Nord Pool, only options with seasonal or yearly contract as underlying is traded. For each contract we estimate the price and delta hedge with different strike price and for two different interest rate, 3% and 5%. The estimated prices and delta hedge are given under.

K	P_Bjerk	P_Benth	delta_Bjerk	delta_Benth
80	31.45	11.76	0.707	0.9455
85	29.04	7.04	0.677	0.8939
90	26.90	2.31	0.64	0.8451
95	24.89	0	0.61	0.7989
100	23.02	0	0.58	0.7553

Table 3.3: Call option prices and delta hedges, seasonal contract, interest rate 3%

K	P_Bjerk	P_Benth	delta_Bjerk	delta_Benth
80	30.30	11.33	0.68	0.68
85	28.02	6.78	0.65	0.626
90	25.92	2.23	0.62	0.57
95	23.97	0	0.59	0.5194
100	22.18	0	0.56	0.4730

Table 3.4: Call option prices and delta hedges, seasonal contract, interest rate 5%

In table 3.3 and 3.4 estimated call option prices for a seasonal contract are given but with different interest rate. First and foremost we see the significant difference in the estimated price. The prices given by Bjerk Sund et al. are much higher compared to Benth et al. In figure 3.10 we might have a possible answer to this. The swap price simulated by Bjerk Sund et al. have a greater volatility than Benth et al., and therefore the price is higher than Benth et al. In addition the price decrease as strike price increase. This is natural, since the estimated mean of each contract is 100. With a strike price significant lower than the mean increase the risk for the seller of the option, and hence the price increase.

Second the delta hedge estimated by Bjerksund et al. is lower than the one estimated by Benth et al., but for each model in decrease as strike price increase. In the same way as for the price, this is natural. The closer strike price is to the expected mean of the contract, the less sensitive the call option price is to a change in the underlying.

And finally, the choice of interest rate seem to play a crucial role. Since the price modeled by Benth et al. is significantly lower than the price given by Bjerksund et al. and as strike price increase, the price estimated by Benth et al. becomes to low.

K	P_Bjerk	P_Benth	delta_Bjerk	delta_Benth
80	34.9	0.24	0.6873	0.9163
85	32.45	0	0.6638	0.8395
90	30.96	0	0.607	0.7692
95	29.18	0	0.6407	0.704
100	27.52	0	0.59	0.6457

Table 3.5: Call option prices and delta hedges, yearly contract, interest rate 3%

K	P_Bjerk	P_Benth	delta_Bjerk	delta_Benth
80	32.54	0.22	0.86	0.55
85	31.29	0	0.62	0.48
90	29.0	0	0.60	0.41
95	27.89	0	0.58	0.36
100	25.96	0	0.56	0.31

Table 3.6: Call option prices and delta hedges, yearly contract, interest rate 5%

In table 3.5 and 3.6 we have estimated call options prices with a yearly contract as underlying. The same features as for a seasonal contract as underlying are observed. The price is significant lower modeled by Benth et al. model than Bjerksund et al. But the price is over all higher than for a seasonal contract as underlying. Indicating as time of delivery increase in the underlying, the bigger risk.

The delta hedge has increased as time of the underlying increased. Indicating the call price is more sensitive to fluctuations in the underlying contract.

As for the call price with a seasonal contract as underlying, the interest rate highly influence the prices.



## Chapter 4

# Final Remarks & Further Research

The previous chapters have introduced two models for the swap-price. The goal was to establish an analysis of the swap price and log-returns given Benth et al. model. Second we compared the swap price and log-returns of the models.

When comparing the "real" swap price to the approximated swap price given by Bjerksund et al. We could see the similarities in the swap price paths, but Bjerksund et al. gave a possibility of greater swap prices, than Benth et al.

All over we have seen the log-returns given by Benth et al. became identical and independent. But if the concern is to establish almost equal swap price paths, then we would have to involve a better approximation. This might be to introduce a forward contract with three independent Brownian Motion, instead of one, as we have used. The given approach in previous section is the author of this thesis choice. Mainly for comparison, since the model by Benth et al. just involves one BM.

When modeling the option, we saw significant difference in the price. This is mainly because of the different price paths as described over. Since the price paths by Bjerksund et al. give a possibility of greater and smaller swap price values, the price of the option will take this into consideration when estimating a price. In addition we saw the great importance the interest rate played.

And finally, the estimations have been under the risk-neutral-measure  $Q$ . If the goal is to compare to real data we must include a risk parameter for each model. The risk parameter is often estimated as the difference between empirical price and estimated price under  $Q$

Further investigation could be to include a jump-process in the forward dynamic. This because we know there occur spikes due to e.g. sudden power limits. Instead of the theoretical price at 100 for each delivery time  $u$ , it would be interesting to include a smoothing algorithm for the forward-price. And

then compare the models with the new  $f(0, u)$  to empirical data. With just one BM, we saw the delivery points would be perfectly correlated, to increase the number, will give different correlation as time points far away would correlate less than for periods, close to each other.

Implementing a stochastic interest rate model, can improve the option price.

And finally compare a HJM-model to a spot-model together with empirical data.

# Appendix A

## Technical Conditions & Proof

### A.1 Some Mathematical Preliminaries

We start out with a basic definition which is well-known but important since it will be our driving process.

**Definition 1.** *Brownian Motion*

A Brownian motion  $W(t)$  is a stochastic process satisfying the following conditions

- $W(0) = 0$
- *Independent increments:* The stochastic variable  $W(v) - W(u)$  is independent of the variable  $W(t) - W(s)$ , where  $v > u \geq t > s \geq 0$
- *Stationary increments:* The distribution to  $W(v) - W(u)$  is only depending on  $v - u$  and not  $v$  and  $u$ .
- *Normal increments:*  $W(v) - W(u) \sim N(0, v - u)$ .

**Definition 2.** *Log-normal variable*

If  $X \sim N(\mu, \sigma^2)$  then  $Y = e^X$  is a log-normal variable with

$$E[Y] = e^{\mu + \frac{\sigma^2}{2}}$$

and

$$\text{Var}(Y) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$$

**Definition 3.** *Probability space*

If  $\Omega$  is a given set, then a  $\sigma$ -algebra  $\mathcal{F}$  on  $\Omega$  is a family  $\mathcal{F}$  of subsets of  $\Omega$  with the following properties:

- $\emptyset \in \mathcal{F}$
- $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$ , where  $F^C = \Omega \setminus F$  is the complement of  $F$  in  $\Omega$

- $A_1, A_2, \dots \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

The pair  $(\Omega, \mathcal{F})$  is called a measurable space.

We can think of the  $\sigma$ -algebra  $\mathcal{F}_t$ , as all possible market information prior and including time  $t$ , e.g. all historical prices of a stock up and including time  $t$ .

**Definition 4. Probability measure**

A probability measure  $P$  on a measurable space  $(\Omega, \mathcal{F})$  is a function  $P : \mathcal{F} \rightarrow [0, 1]$ , such that

- $P(\emptyset) = 0$
- $P(\Omega) = 1$
- if  $A_1, A_2, \dots \in \mathcal{F}$  and  $\{A_i : i \in [1, \infty]\}$  is disjoint then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

**Definition 5. Probability space**

The trippel  $(\Omega, \mathcal{F}, P)$  is called a probability space. It is called a complete probability space if  $\mathcal{F}$  contains all subsets  $G$  of  $\Omega$  with  $P$ -outer measure zero i.e. with

$$P^*(G) := \inf\{P(F); F \in \mathcal{F}, G \subset F\} = 0$$

**Definition 6. Martingale**

An  $n$ -dimensional stochastic process  $\{M(t) : t \geq 0\}$  on a probability space  $(\Omega, \mathcal{F}, P)$  is called a martingale with respect to a filtration  $\mathcal{F}_{t \in [0, T]}$  and  $P$  if

- $M(t)$  is  $\mathcal{F}_t$ -measurable  $\forall t$
- $E[|M(t)|] < \infty \forall t$
- $E[M(s) | \mathcal{F}_t] = M(t), \forall t \leq s$

**Definition 7. Square-Integrable Martingale**

A right-continuous martingale  $X$  is square-integrable if  $E[X_t^2] < \infty$  for every  $t \geq 0$

If a process i.e. a stock price process, is a martingale, then we are not able to say anything with a 100% certainty about the future. Hence we do not allow for inside information. Specially if the stock price process is square-integrable martingale, we are restricted against any stock price being infinite.

**Definition 8. Arbitrage**

Arbitrage is to profit without taking any risk. There is a positive probability that you can earn money without having a positive probability of losing the money.



## A.2 Swap dynamics

- The random field  $(t, T_1, T_2, \omega) \rightarrow \Sigma(t, T_1, T_2, \omega)$  is  $B \times \mathcal{F}$  - measurable, where  $B$  is the Borel  $\sigma$ -algebra on  $\{(t, T_1, T_2) \in [0, T]^3 : t \leq T_1 < T_2\}$
- For all  $(T_1, T_2)$  such that  $0 < T_1 \leq T_2 \leq T$ ,  $[0, T_1] \ni t \rightarrow \Sigma(t, T_1, T_2)$  is progressively measurable, and

$$E\left[\int_0^{T_1} \Sigma^2(t, T_1, T_2) dt\right] < \infty$$

## A.3 Forward dynamics

- The random field  $(t, u, \omega) \rightarrow \sigma(t, u, \omega)$  is  $B \times \mathcal{F}$  - measurable, where  $B$  is the Borel  $\sigma$ -algebra on  $\{(t, u) \in [0, T]^2 : t \leq T\}$
- For all  $0 < u < T$ ,  $[0, u] \ni s \rightarrow \sigma(s, u)$  is progressively measurable, and

$$E\left[\int_0^u \sigma^2(s, u) ds\right] < \infty$$

## A.4 Settlement at maturity

Let  $\hat{w}$  be given by

$$\hat{w}(u_i, T_1, T_2) = \frac{w(u_i)}{\int_{T_1}^{T_2} w(s) ds}$$

where

$$w(u_i) = \begin{cases} 1, & \text{settlement at maturity} \\ e^{-ru_i}, & \text{settlement continuously during the delivery period} \end{cases}$$

We are only concerning with settlement at maturity, we get

$$\hat{w}(u_i, T_1, T_2) = \frac{1}{T_2 - T_1}$$

## A.5 Forward and Swap dynamics

For the dynamics in (2.4) and (2.3), suppose the following (in addition to the former)

- $[0, T_1] \times \Omega \times [T_1, T_2] \ni ((s, \omega), u) \rightarrow \hat{w}(u; T_1, T_2) \sigma(s, u, \omega)$  is jointly progressively measurable and measurable with respect to  $B([T_1, T_2])$
- $(u, \omega) \rightarrow \hat{w}(u; T_1, T_2) \sigma(s, u, \omega)$  is  $B([0, T_1]) \times \mathcal{F}$ -measurable
- The following integrability holds:

$$E\left[\int_0^{T_1} \int_{T_1}^{T_2} \hat{w}^2(u; T_1, T_2) \sigma^2(s, u) du ds\right] < \infty \quad (\text{A.1})$$

- The initial forward curve  $[T_1 - T_2] \ni t \rightarrow f(0, t) \mathbf{B}([T_1, T_2])$ -measurable and

$$\int_{T_1}^{T_2} \hat{w}(t; T_1, T_2) f^2(0, t) dt < \infty \quad (\text{A.2})$$

## A.6 Proof of Black -76 call option

**Proof 1.** Let the dynamic of the forward contract be given by,

$$\begin{aligned} df(T, \tau) &= \sigma(T, \tau) f(T, \tau) dW(T) \\ f(T, \tau) &= f(t, \tau) e^{-\frac{1}{2} \int_t^T \sigma^2(s, \tau) ds + \sqrt{\int_t^T \sigma^2(s, \tau) ds} X} \\ \text{where } X &\sim \mathbf{N}(0, 1) \end{aligned}$$

We want to derive the price of a call option with the forward contract as underlying,

$$\begin{aligned} P(t, T, K, \tau) &= e^{-r(T-t)} E_Q[(f(T, \tau) - K)^+ | \mathcal{F}_t] \\ &= e^{-r(T-t)} E[(e^{\ln(f(t, \tau)) - \frac{1}{2} \int_t^T \sigma^2(s, \tau) ds + \sqrt{\int_t^T \sigma^2(s, \tau) ds} X} - K)^+] \end{aligned}$$

Let  $z = \ln(f(t, \tau)) - \frac{1}{2} \int_t^T \sigma^2(s, \tau) ds$  and  $q(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

$$\begin{aligned} &= e^{-r(T-t)} E[(e^{z + \sqrt{\int_t^T \sigma^2(s, \tau) ds} X} - K)^+] \\ &= e^{-r(T-t)} \int_{z + \sqrt{\int_t^T \sigma^2(s, \tau) ds}}^{\infty} (e^{z + \sqrt{\int_t^T \sigma^2(s, \tau) ds} y} - K) q(y) dy \\ &= e^{-r(T-t)} \int_{\frac{\ln(K) - z}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}}}^{\infty} e^{z + \sqrt{\int_t^T \sigma^2(s, \tau) ds} y} q(y) dy - e^{-r(T-t)} K \int_{\frac{\ln(K) - z}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}}}^{\infty} q(y) dy \end{aligned}$$

We derive the first integral:

$$\begin{aligned} &= e^{-r(T-t)} \int_{\frac{\ln(K) - z}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}}}^{\infty} e^{z + \sqrt{\int_t^T \sigma^2(s, \tau) ds} y} q(y) dy \\ &= e^{-r(T-t)} \int_{\frac{\ln(K) - z}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}}}^{\infty} \frac{1}{2\pi} e^{\ln(f(t, \tau)) - \frac{1}{2} \int_t^T \sigma^2(s, \tau) ds + \sqrt{\int_t^T \sigma^2(s, \tau) ds} y - y^2/2} dy \\ &= e^{-r(T-t)} e^{\ln(f(t, \tau)) - \frac{1}{2} \int_t^T \sigma^2(s, \tau) ds + \frac{1}{2} \int_t^T \sigma^2(s, \tau) ds} \int_{\frac{\ln(K) - z}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}}}^{\infty} e^{-\frac{1}{2} \left( y - \sqrt{\int_t^T \sigma^2(s, \tau) ds} \right)^2} dy \\ &= e^{-r(T-t)} e^{\ln(f(t, \tau))} \int_{\frac{\ln(K) - z}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}} - \sqrt{\int_t^T \sigma^2(s, \tau) ds}}^{\infty} e^{-\frac{1}{2} v^2} dv \end{aligned}$$

Where we use substitution  $v = y - \sqrt{\int_t^T \sigma^2(s, \tau) ds}$ , hence we get a normal distributed variable.

$$\begin{aligned}
&= e^{-r(T-t)} e^{\ln(f(t, \tau))} \int_{-\infty}^{\sqrt{\int_t^T \sigma^2(s, \tau) ds} - \frac{\ln(K) - z}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}}} q(v) dv \\
&= e^{-r(T-t)} e^{\ln(f(t, \tau))} \Phi \left( \sqrt{\int_t^T \sigma^2(s, \tau) ds} - \frac{\ln(K) - z}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}} \right) \\
&= e^{-r(T-t)} e^{\ln(f(t, \tau))} \Phi \left( \sqrt{\int_t^T \sigma^2(s, \tau) ds} - \frac{\ln(K/f(t, \tau)) + \frac{1}{2} \int_t^T \sigma^2(s, \tau) ds}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}} \right) \\
&= e^{-r(T-t)} f(t, \tau) \Phi \left( \frac{\int_t^T \sigma^2(s, \tau) ds + \ln(f(t, \tau)/K) - \frac{1}{2} \int_t^T \sigma^2(s, \tau) ds}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}} \right) \\
&= e^{-r(T-t)} f(t, \tau) \Phi \left( \frac{\ln(f(t, \tau)/K) + \frac{1}{2} \int_t^T \sigma^2(s, \tau) ds}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}} \right)
\end{aligned}$$

The last integral will be:

$$\begin{aligned}
e^{-r(T-t)} K \int_{\frac{\ln(K) - z}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}}}^{\infty} q(y) dy &= e^{-r(T-t)} K \Phi \left( \frac{z - \ln(K)}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}} \right) \\
&= e^{-r(T-t)} K \Phi \left( \frac{\ln(f(t, \tau)/K) - \frac{1}{2} \int_t^T \sigma^2(s, \tau) ds}{\sqrt{\int_t^T \sigma^2(s, \tau) ds}} \right)
\end{aligned}$$



# Appendix B

## R-code

### B.1 Swap price modeling

```
1
2 library(tseries)
3 library(fCalendar)
4 library(fUtilities)
5
6
7 length          <- 5                      #1,2,3,4,5
                        #Defining the length of the forward contract,
8
9
10 TTD             <- c(4 ,20, 168, 480, 730)
                        #Time to Delivery, Tradingdays, day, week, month, season
11               , year
12 regular         <- c(1 , 7,  28, 120, 365)
13 T1              <- TTD[length]
                        #Deliveryperiod, "regular days"
14 T2              <- TTD[length] + regular[length]
15
16 days           <- T2-T1
17
18 sim            <- 10000
19
20
21 #####
22 ## Benth and Koekebakker ##
23 #####
24
25 m              <- 6                      #2,5,6
                        #Defining the volatility model
26
27 swap           <- matrix(NA, T1, sim)
28 mswap          <- 0*(1:T1)
29
30 return         <- NA*(1:(T1-1))
31 norm_return    <- NA*(1:(T1-1))
32
33 volatility      <- matrix(NA, T1, days)
```

```

34 volatility[1,] <- 0.5/250
35
36 a          <- c(0, 0.680, 0, 0, 0.781, 0.604)
           #Parameters given in Benth et. al (2008)
37 b          <- c(0, 0.784, 0, 0, 2.667, 2.848)
38 c          <- c(0, 0, 0, 0, 0.190, 0.161)
39 d          <- c(0, 0, 0, 0, 0.066, 0.018)
40 f          <- c(0, 0, 0, 0, -0.179, -0.065)
41
42
43 #Estimating the volatilities
44 for(k in 2:T1){
45     for(j in 1:days){
46         volatility[k,j] <- (integrate(intSigma2, k-1, k, j)
47                               $value)/250
48     }
49 }
50
51 #Integration- function, integrating over the desired time-inkrement
52 intSigma2    <- function(k, j){
53     if(m == 2){
54         ((a[m]*exp(-b[m]*((T1+j-1)-k)))^2)
55
56         #only
57         maturity effect, very simple, E2
58
59     }else if(m == 5){
60         (((c[m]) + (d[m]*sin((2*pi*k)/365))-(f[m]*cos((2*pi
61             *k)/365))))^2
62
63         #only season effect
64
65     }else if(m == 6){
66         ((a[m]*exp(-b[m]*((T1+j-1)-k)))+ c[m] + (d[m]*sin
67             ((2*pi*k)/365))-(f[m]*cos((2*pi*k)/365))))^2
68
69         #season and maturity effect, E6
70     }
71 }
72
73 #Modeling the forward dynamic
74 the_forward <- function(eps3){
75     forward <- matrix(NA, T1, days)
76     forward[1,] <- 100
77
78     swap_deliver <- 0*(1:T1)
79
80     for(k in 2:T1){
81         forward[k,] <- forward[k-1,] * exp(((1/2)*
82             volatility[k,]) + (sqrt(volatility[k,])*eps3[k])
83             )
84     }
85     swap_deliver <- rowSums(forward)*(1/days)
86 }
87
88 #Calculating the swap price with sum of forwards

```

```

80 for(i in 1:sim){
81     eps3 <- rnorm(T1, 0, 1)
82     swap[, i] <- the_forward(eps3)
83 }
84
85 mswap <- rowMeans(swap)
86 return[1] <- log(mswap[1]/mswap[2])
87
88 #Calculating daily log-returns
89 for(k in 2:(T1-1)){
90     return[k] <- log(mswap[k]/mswap[k-1])
91 }
92
93
94 #The simulated means of each time t, before delivery
95
96 norm_return <- (return-mean(return))/sd(return)
97
98 analyze <- matrix(0,2,7)
99 analyze[1,1] <- 'Trading_days'
100 analyze[1,2] <- 'Volatility_Model'
101 analyze[1,3] <- 'Mean_Log-Return'
102 analyze[1,4] <- 'Annual_Standard_Deviation_in_log-return'
103 analyze[1,5] <- 'Min_log-return'
104 analyze[1,6] <- 'Max_log-return'
105 analyze[1,7] <- 'Difference_price_log-return'
106
107 analyze[2,1] <- T1
108 analyze[2,2] <- m
109 analyze[2,3] <- mean(mswap)
110 analyze[2,4] <- sd(return)*sqrt(250)
111 analyze[2,5] <- min(return)
112 analyze[2,6] <- max(return)
113 analyze[2,7] <- max(return)-min(return)
114
115 analyze
116
117 sd(mswap)*sqrt(250)
118
119 skewness(return)
120 kurtosis(return)
121
122 adf.test(mswap)
123 adf.test(return)
124
125 #####
126 ## Bjerksund, Rasmussen and Stensland Approach ##
127 #####
128
129 swap_B <- matrix(NA, T1, sim)
130 swap_B[1,] <- 100
131 mswap_B <- NA*(1:T1)
132
133 return_B <- NA*(1:(T1-1))
134 norm_return_B <- NA*(1:(T1-1))
135
136 volatility_B <- NA*(1:T1)

```

```

137
138 b_B <- 0.1406
139 a_B <- b_B*0.60669
140 c_B <- 0.21169
141
142
143 #Given the estimations for the volatility function
144 int_B1 <- function(k){
145   ((T2+b_B-k)*(log(T2+b_B-k))^2)-(2*(T2-b_B-k)*log(T2+b_B-k)*
146     log(T1+b_B-k))
147   +(4*a_B*log(T2-T1)*log((T1+b_B-k)/(T2-T1)))- ( 2*(T2-T1)*
148     dilog((T2+b_B-k)/(T2-T1)) )
149   +((T2+b_B-k)* (log(T1+b_B-k))^2 - 2*(T2-T1) )
150   -((T2+b_B-k+1)*(log(T2+b_B-k+1))^2)-(2*(T2-b_B-k+1)*log(T2+
151     b_B-k+1)*log(T1+b_B-k+1))
152   +(4*a_B*log(T2-T1)*log((T1+b_B-k+1)/(T2-T1)))- ( 2*(T2-T1)
153     )*dilog((T2+b_B-k+1)/(T2-T1)) )
154   +((T2+b_B-k+1)* (log(T1+b_B-k+1))^2 - 2*(T2-T1) )
155 }
156
157 dilog <- function(x){
158   di <- 0
159   if(x<= 1){
160     for(j in 1: 1000){
161       di <- di + ((x-1)^j/j^2)
162     }
163   }else{
164     di <- -(1/2)*(log(x))^2
165     for(j in 1:1000){
166       di <- di + (((1/x)-1)^j/j^2)
167     }
168   }
169   di
170 }
171
172 int_B2 <- function(k){
173   ((T2+b_B-k)*log(T2+b_B-k))
174   -((T1+b_B-k)*log(T1+b_B-k))
175   -((T2+b_B-k+1)*log(T2+b_B-k+1))
176   +((T2+b_B-k+1)*log(T1+b_B-k+1))
177 }
178
179 #Calculating the volatility , "big sigma"
180 for(k in 1:T1){
181   volatility_B[k] <- (((a_B^2/(T2-T1)^2))*int_B1(k) + (((2*a_
182     B*c_B)/(T2-T1))*int_B2(k)) + (c_B *(k-k+1)))/250
183 }
184
185 #T1*sim standard normal variables
186 eps4 <- matrix(rnorm(T1*sim, 0, 1), T1, sim)
187
188 #estimates the swap price sim times

```



```

189 for(k in 2:T1){
190     swap_B[k,] <- swap_B[k-1,] * exp((-1/2)* volatility_B[
      k] + sqrt(volatility_B[k])*eps4[k,] )
191 }
192
193 mswap_B <- rowMeans(swap_B)
194 return_B[1] <- log(mswap_B[1]/mswap_B[2])
195
196 #Calculating the log-returns
197 for(k in 2:(T1-1)){
198     return_B[k] <- log(mswap_B[k]/mswap_B[k-1])
199 }
200
201
202 norm_return_B <- (return_B-mean(return_B))/sd(return_B)
203
204 analyze_B <- matrix(0,2,7)
205 analyze_B[1,1] <- 'Trading_days'
206 analyze_B[1,2] <- 'Volatility_Model'
207 analyze_B[1,3] <- 'Mean_Log-Return'
208 analyze_B[1,4] <- 'Annual_Standard_Deviation_in_log-return'
209 analyze_B[1,5] <- 'Min_log-return'
210 analyze_B[1,6] <- 'Max_log-return'
211 analyze_B[1,7] <- 'Difference_price_log-return'
212
213 analyze_B[2,1] <- T1
214 analyze_B[2,2] <- m
215 analyze_B[2,3] <- mean(return_B)
216 analyze_B[2,4] <- sd(return_B)*sqrt(250)
217 analyze_B[2,5] <- min(return_B)
218 analyze_B[2,6] <- max(return_B)
219 analyze_B[2,7] <- max(return_B)-min(return_B)
220
221 analyze_B
222
223 sd(mswap_B)*sqrt(250)
224
225 skewness(return_B)
226 kurtosis(return_B)
227
228 adf.test(mswap_B)
229 adf.test(return_B)
230
231 #####
232 ## Comparison ##
233 #####
234 #Plot of simulated qq-plot
235 #par(mfrow = c(2,2))
236 tt <- qqnorm(rev(norm_return), main = "Season_contract", plot
      = FALSE)
237 dd <- qqnorm(rev(norm_return_B), col = 'red', plot = FALSE)
238
239 plot(tt, pch = 15, main = "Yearly_contract", xlab = "", ylab = "")
240 lines(dd, type = 'p', col = 'red', pch = 16)
241 qqline(norm_return_B)
242 legend("topleft", pch = 15:16, c("Benth_et_al.", "Bjerk Sund_et_al."),
      , col = c("black", "red"))

```

```

243
244
245 #Plot of simulated swap prices, both by in Benth and Bjerksund case
246 par(mfrow = c(2,1))
247 plot(rev(mswap), type = 'l', main = "Seasonly_contract", xlab = "Time_To_Maturity", ylab = "Swap_price",
248 lwd = 3, ylim = range(mswap, swap[,1], swap[,2], swap[,3], swap[,4],
swap[,5], mswap_B, swap_B[,1], swap_B[,2], swap_B[,3], swap_B
[,4], swap_B[,5] ))
249 for(i in 1:5){
250     lines(rev(swap[, (3*i)]), col = 1)
251 }
252
253 lines(rev(mswap_B), col = 2)
254 for(i in 1:5){
255     lines(rev(swap_B[, i]), col = 2)
256 }
257 legend("topright", lty = 1, c("Benth_et_al.", "Bjerksund_et_al."),
col = c("black", "red"))
258
259
260 #Plot of simulated log-returns prices, both by in Benth and
Bjerksund case
261 plot(rev(mreturn), type = 'l', main = "Yearly_contract", xlab = "Time_To_Maturity", ylab = "Log-returns", ylim = range(rev(
mreturn), rev(mreturn_B)))
262 lines(rev(mreturn_B), col = 'red')
263 legend("topright", lty = 1, c("Benth_et_al.", "Bjerksund_et_al."),
col = c("black", "red"))

```

## B.2 Option modeling

```

1  library(tseries)
2  library(fCalendar)
3
4  length          <- 5                                #4,5                                #
Defining the length of the forward contract, day, week, month,
season, year
5  excer           <- 0
6  strike          <- 100
7  r               <- 0.05/250
8
9  TTD             <- c(4, 25, 168, 480, 730)
#Time to Delivery, Tradingdays, day, week, month, season
, year
10 regular         <- c(1, 7, 28, 120, 365)
11
12 T1              <- TTD[length]
#Deliveryperiod, "regular days"
13 T2              <- TTD[length] + regular[length]
14
15 days            <- T2-T1
16
17
18 if(length == 4){
19     excer         <- T1-12
20 }else if(length == 5){

```

```

21         excer                                     <- T1-1
22     }
23
24     sim                                             <- 10000
25
26     #####
27     ## Benth and Koekebakker ##
28     #####
29
30     m                                             <- 6
31                                     #Defining the volatility model
32     a                                             <- c(0, 0.680, 0, 0, 0.781, 0.604)
33                                     #Parameters given in Benth et. al (2008), to be used in
34                                     the volatility function
35     b                                             <- c(0, 0.784, 0, 0, 2.667, 2.848)
36     c                                             <- c(0, 0, 0, 0, 0.190, 0.161)
37     d                                             <- c(0, 0, 0, 0, 0.066, 0.018)
38     f                                             <- c(0, 0, 0, 0, -0.179, -0.01)
39
40     swap                                         <- 0*(1:sim)
41     delta                                       <- 0*(1:sim)
42
43     volatility_excer                             <- 0*(1:days)
44
45     price                                       <- 0
46     strike_swap                               <- 0*(1:sim)
47     mdelta                                     <- 0
48
49     for(j in 1:days){
50         volatility_excer[j]                     <-(integrate(intSigma2, 0, excer, j
51             )$value)/250
52     }
53
54     #Integration- function, for the forward dynamic, integrating with
55     respect to k, and j is a given parameter
56     intSigma2                                   <- function(k, j){
57         (( (a[m]*exp(-b[m]*((T1+j-1)-k)))+ c[m] + (d[m]*sin((2*pi*k
58             )/365))-(f[m]*cos((2*pi*k)/365))))^2)          #season and
59             maturity effect, E6
60     }
61
62     #Modelling the swap dynamic, first; for each day ahead of the
63     delivery-time, I approximate the integral by the algorithm given
64     in Benth et. al p. 1137
65     the_forward                                 <- function(eps2){
66         forward_price                           <- 0*(1:days)
67         delta_test                             <- 0*(1:days)
68
69         for(i in 1:days){
70             forward_price[i]                   <- 100 * exp((( -1/
71                 2)*volatility_excer[i]) + (sqrt(volatility_excer
72                 [i])*eps2))

```

```

67         delta_test[i] <- exp(((1/2)*volatility -
68             excer[i]) + (sqrt(volatility_excer[i])*eps2))
69     }
70     s1 <- sum(forward_price)*(1/days)
71     s2 <- sum(delta_test)*(1/days)
72     c(s1,s2)
73 }
74
75
76
77
78 #Calculating the swap price with sum of forwards
79 for(i in 1:sim){
80     eps2 <- rnorm(1, 0, 1)
81     go <- the_forward(eps2)
82     swap[i] <- go[1]
83     delta[i] <- go[2]
84 }
85
86 for(i in 1:sim){
87     if((swap[i]-strike)<0){
88         strike_swap <- 0
89     }else{
90         strike_swap <- swap[i]-strike
91     }
92 }
93
94 price <- mean(strike_swap)*exp(-r*excer)
95 delta <- mean(delta)*exp(-r*excer)
96
97 #####
98 ## Bjerk sund, Stensland and Rasmussen ##
99 #####
100
101 option_vol_B <- 0
102 price_B <- 0
103 delta_B <- 0
104
105 b_B <- 0.1406
106 a_B <- b_B*0.60669
107 c_B <- 0.20331
108
109 option_vol_B <- (((a_B^2/(T2-T1)^2)*int_B1_option()) + (((2*a_B*
110     c_B)/(T2-T1))*int_B2_option()) + (c_B *excer))/250
111
112 dilog <- function(x){
113     di <- 0
114     if(x<= 1){
115         for(j in 1: 1000){
116             di <- di + ((x-1)^j/j^2)
117         }
118     }else{
119         di <- -(1/2)*(log(x))^2
120         for(j in 1:1000){
121             di <- di + (((1/x)-1)^j/j^2)

```

```

122     }
123   }
124   di
125 }
126
127
128 int_B1_option <- function() {
129   ((T2+b_B-excer)*(log(T2+b_B-excer))^2)-(2*(T2-b_B-excer)*
130     log(T2+b_B-excer)*log(T1+b_B-excer))
131   +(4*a_B*log(T2-T1)*log((T1+b_B-excer)/(T2-T1)))- ( 2*(T2-
132     T1)*dilog((T2+b_B-excer)/(T2-T1)) )
133   +((T2+b_B-excer)* (log(T1+b_B-excer))^2 - 2*(T2-T1) )
134   -((T2+b_B)*(log(T2+b_B))^2)-(2*(T2-b_B)*log(T2+b_B)*log(T1+
135     b_B))
136   +(4*a_B*log(T2-T1)*log((T1+b_B)/(T2-T1)))- ( 2*(T2-T1)*
137     dilog((T2+b_B)/(T2-T1)) )
138   +((T2+b_B)* (log(T1+b_B))^2 - 2*(T2-T1) )
139 }
140
141 int_B2_option <- function() {
142   ((T2+b_B-excer)*log(T2+b_B-excer))
143   -((T1+b_B-excer)*log(T1+b_B-excer))
144   -((T2+b_B)*log(T2+b_B))
145   +((T2+b_B)*log(T1+b_B))
146 }
147
148 d1_B <- function() {
149   ( log(100/strike)+ (0.5 * option_vol_B)) / (sqrt(option_vol
150     _B))
151 }
152
153 d2_B <- function() {
154   ( log(100/strike)- (0.5 * option_vol_B)) / (sqrt(option_vol
155     _B))
156 }
157
158 #The option price formula
159 n_d1 <- pnorm(d1_B())
160 n_d2 <- pnorm(d2_B())
161
162 price_B <- exp(-r*excer)*((100*n_d1) - (strike*n_d2))
163 delta_B <- exp(-r*excer)*n_d1
164
165
166 price_B
167 delta_B
168
169 price
170 delta

```



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